DIRECTORATE OF DISTANCE EDUCATION
UNIVERSITY OF NORTH BENGAL

MASTER OF SCIENCES- MATHEMATICS

SEMESTER -III

DISCRETE MATHEMATICS

DEMATH3OLEC4

BLOCK-1

UNIVERSITY OF NORTH BENGAL

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FOREWORD

The Self Learning Material (SLM) is written with the aim of providing simple and organized study content to all the learners. The SLMs are prepared on the framework of being mutually cohesive, internally consistent and structured as per the university's syllabi. It is a humble attempt to give glimpses of the various approaches and dimensions to the topic of study and to kindle the learner's interest to the subject

We have tried to put together information from various sources into this book that has been written in an engaging style with interesting and relevant examples. It introduces you to the insights of subject concepts and theories and presents them in a way that is easy to understand and comprehend.

We always believe in continuous improvement and would periodically update the content in the very interest of the learners. It may be added that despite enormous efforts and coordination, there is every possibility for some omission or inadequacy in few areas or topics, which would definitely be rectified in future.

We hope you enjoy learning from this book and the experience truly enrich your learning and help you to advance in your career and future endeavours.

DISCRETE MATHEMATICS

BLOCK-1

BLOCK-2

BLOCK- 1 DISCRETE MATHEMATICS

Discrete mathematics is the study of mathematical structures that are countable or otherwise distinct and separable. Examples of structures that are discrete are combinations, graphs, and logical statements. Discrete structures can be finite or infinite. Discrete mathematics is in contrast to **continuous mathematics**, which deals with structures which can range in value over the real numbers, or have some non-separable quality.

Discrete structures can be counted, arranged, placed into sets, and put into ratios with one another. Although discrete mathematics is a wide and varied field, there are certain rules that carry over into many topics. The concept of independent events and the rules of product, sum, and PIE are shared among combinatorics, set theory, and probability.

Discrete mathematics concerns itself mainly with finite collections of discrete objects. With the growth of digital devices, especially computers, discrete mathematics has become more and more important.

UNIT 1: SET THEORY

STRUCTURE

1.0 OBJECTIVES

- What is a Set?
- What are the different ways to represent the Sets?
- What are the different types of Sets?
- Operation on Sets
- Representation of Sets using Venn Diagram

1.1 WHAT IS A SET?

SET: It is a collection of things.

Things like what we carry in our school/college bag as books, pen, and geometry set, Tiffin, Water bottle, napkin, etc.When we write these things in curly bracket like below, it represent the Set& Setis always denoted by Capital Letters like A, B, C, etc.

S = {Books, pen, geometry set, tiffin, water bottle, napkin}

Example: Set of Whole numbers: - W = {0, 1, 2, 3…}

Set of Alphabets: - A = {a, b, c… z}

Each member like in above example i.e. 0,1, 2 or a, b c, are known as ***elements*** of the Set& is represented with lowercase letters it is denoted by '∈'.

So a ∈A – a is an element of A

a ∉W - a is not an element of W

<u>Five ways to describe set:</u>

1. Describing the ***properties*** of the members of the set
2. Describing by ***listing*** its elements
3. Describing by its ***characteristics function***
    a. µA(x) = 1 if x∈ A
    b. µA(x) = 0 if x∉ A
4. Describing by ***recursive formula***
5. Describe by an ***operation***(such as union, intersection, complement, etc.)

Example: Describe the set containing natural numbers up to 5

Let N denote the set then we can describe N in following ways

1. N= {x | x is natural number less than or equal to 5}
2. N ={1,2,3,4,5}
3. $\mu_N(x) = \begin{cases} 1 & for\ x = 1,2,\ldots,5 \\ 0 & otherwise \end{cases}$
4. $N = \{x_{i+1} = x_i + 1,\ i = 0,1,\ldots,4\ where\ x_0 = 0\}$
5. This type will be discussed ahead in the section 'Operation on Set'.

<u>CONCEPTS RELATED TO SETS:</u>

1. **Subset:** Let A and B be two sets such that

   A is a subset of B and it is represented as **'$A \subseteq B$'** if every element of A is an element of B

   A is a proper subset of B and it is represented as **'$A \subset B$'** if A is a subset of B & atleast one element of B which is not there in A.

   <u>Properties related to Subset:</u>

   1. $A \subseteq A$
   2. If $A \subseteq B$ & $B \subseteq C$, then $A \subseteq C$
   3. If $A \subset B$ & $B \subset C$ then $A \subset C$
   4. If $A \subseteq B$ & $A \not\subseteq C$, then $B \not\subseteq C$ where $\not\subseteq$ means is not contained in.

2. **Equal Sets:** Two sets are equal when $A \subseteq B$ & $B \subseteq A$ that is $A=B$

3. **Empty Set or Null Set**: A set containing no element & denoted by $\emptyset$. The empty set is a subset of every set.

4. **Singleton :** A set containing one element

5. **Universal Set**: A set that contains everything

Check Your Progress 1

1. Define Set and explain different ways to describe the sets.

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2. Explain the following concepts with examples

   a. Equal Sets

   b. Universal Set

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1.2 OPERATION OF SET

1. **Complement** : Suppose U is the universal set & A be any subset of U, so $\bar{A}$ ***(absolute complement of A)***is {x | x ∉ A} or {x| x∉U and x ∈ A}

   A & B are two sets then relative complement of A with respect to B is

   B – A ={x | x **∈B** and x ∉A}

   From above explanation we can drive the following

   a. The complement of Universal Set U is $\overline{U} = \emptyset$
   b. The complement of Empty Set ∅ is $\overline{\emptyset} = U$

Example: Let U = {1, 2, 3, 4, 5, 6, 7, 8, 9}, A = {1, 2, 3, 4} and B = {2, 4, 6, 8}.

(i) Find A'

(ii) Find B'

Solution:

(i) A' = U - A

= {1, 2, 3, 4, 5, 6, 7, 8, 9} - {1, 2, 3, 4}

= {5, 6, 7, 8, 9}

(ii) B' = U - B

= {1, 2, 3, 4, 5, 6, 7, 8, 9} - {2, 4, 6, 8}

= {1, 3, 5, 7, 9}

2. **Union of Sets:** It is denoted by ∪implies that it contain all the elements of respective set considered in Union.

   Let A & B be two sets then there union is represented as

   $$\boxed{\mathbf{A \cup B = \{x \mid x \in A \text{ or } x \in B \text{ or both}\}}}$$

3. **Intersection of two sets: -**It is denoted by ∩implies that it contain all the common elements of respective set considered in intersection.
   Let A & B be two sets then there union is represented as

   $$\boxed{\mathbf{A \cap B = \{x \mid x \in A \text{ or } x \in B\}}}$$

<u>Properties related to Union & Intersection of two sets:</u>

	UNION	INTERSECTION
Idempotent	$A \cup A = A$	$A \cap B$
Commutative	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Associative	$A \cup (B \cup C) = (A \cup B) \cup C$	$A \cap (B \cap C) = (A \cap B) \cap C$

4. **Symmetrical Difference:** When the elements of the set belong to either one set or other set but not both which are considered for the operation. It is denoted by Δ.

   Let A & B be the two sets so the symmetrical difference of both sets is

$A \Delta B = \{x \mid x \in A$ or $x \in B$, but not both$\}$

5. **Disjoint Set:** They does not have common elements & represented as $A \cap B = \emptyset$

Let us explore few ***theorems*** on the basis of above operations

Theorem 1: Distributive Law

Let A, B & C be three sets then,

$$\boldsymbol{C \cap (A \cup B) = (C \cap A) \cup (C \cap A)}$$

$$\boldsymbol{C \cup (A \cap B) = (C \cup A) \cup (C \cup A)}$$

Example: Let us explain the above properties with the help of example

Let A = {1, 2, 3, 4}, B = {3,4,5,6}, C = {6,7,8} and Universal Set = U = {1,2,3,4,5,6,7,8,9,10}

1. Commutative Law: $A \cap B = B \cap A$

$A \cap B = \{1, 2, 3, 4\} \cap \{3,4,5,6\}$

$= \{3, 4\}$

$\therefore A \cap B = B \cap A$

2. Associative Law: $(A \cap B) \cap C = A \cap (B \cap C)$

$A \cap B = \{3, 4\}$

$(A \cap B) \cap C = \{3, 4\} \cap \{6,7,8\} = \{\} = \emptyset$

$B \cap C = \{3, 4, 5, 6\} \cap \{6,7,8\} = \{6\}$

$A \cap (B \cap C) = \{1, 2, 3, 4\} \cap \{6\} = \{\} = \emptyset$

$\therefore (A \cap B) \cap C = A \cap (B \cap C)$

3.Law of $\emptyset$ and U: $\emptyset \cap A = \emptyset$, $U \cap A = A$

In intersection, we have all common elements

Since $\emptyset$ has no elements , there will be no common element between $\emptyset$ and A. Therefore, intersection of $\emptyset$ and A will be $\emptyset$

$\emptyset \cap A = \{\} \cap \{1, 2, 3, 4\}$

$= \{\}$

$U \cap A = A$

Since U has all the elements, the common elements between U and A will be all the elements of set A

Therefore, intersection of U and A will be A

$U \cap A = \{1, 2, 3, 4, 5, 6, 7, 8, 9,10\} \cap \{1, 2, 3, 4\}$

$= \{1, 2, 3, 4\} = A$

$\therefore U \cap A = A$

4.Idempotent Law : $A \cap A = A$

$A \cap A = \{1, 2, 3, 4\} \cap \{1, 2, 3, 4\}$

$= \{1, 2, 3, 4\} = A$

$\therefore A \cap A = A$

5. Distributive Law i. e. $\cap$ distributes over $\cup$: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$B \cup C = \{3,4,5,6\} \cup \{6,7,8\} = \{3,4,5,6, 7, 8\}$

$A \cap (B \cup C) = \{1, 2, 3, 4\} \cap \{3,4,5,6, 7, 8\}$

$= \{3, 4\}$

$A \cap B = \{1, 2, 3, 4\} \cap \{3,4,5,6\} = \{3, 4\}$

$A \cap C = \{1, 2, 3, 4\} \cap \{6,7,8\} = \{\}= \emptyset$

$(A \cap B) \cup (A \cap C) = \{3, 4\} \cup \emptyset = \{3, 4\}$

$\therefore A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Distributive Law i. e. $\cup$ distributes over $\cap$: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$(B \cap C) = \{3, 4, 5, 6\} \cap \{6,7,8\} = \{6\}$

$A \cup (B \cap C) = \{1, 2, 3, 4\} \cup \{6\} = \{1, 2, 3, 4, 6\}$

$A \cup B = \{1, 2, 3, 4\} \cup \{3, 4, 5, 6\} = \{1, 2, 3,4, 5, 6 \}$

$A \cup C = \{1, 2, 3, 4\} \cup \{6,7,8\}= \{1, 2, 3, 4, 6, 7, 8\}$

$(A \cup B) \cap (A \cup C) = \{1, 2, 3, 4, 5, 6 \} \cap \{1, 2, 3, 4, 6, 7, 8\}$

$= \{1, 2, 3, 4, 6\}$

$\therefore A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Theorem 2:DeMorgan's Law

Let A & B be the two sets then

1. $\overline{A \cup B} = \overline{A} \cap \overline{B}$

2. $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Proof: DeMorgan's Law 1:

Let P = (A U B)' and Q = A' ∩ B'

Let x be an arbitrary element of P then x ∈ P ⇒ x ∈ (A U B)'

⇒ x ∉ (A U B)

⇒ x ∉ A and x ∉ B

⇒ x ∈ A' and x ∈ B'

⇒ x ∈ A' ∩ B'

⇒ x ∈ Q

Therefore, P ⊂ Q …………….. (i)

Again, let y be an arbitrary element of Q then y ∈ Q ⇒ y ∈ A' ∩ B'

⇒ y ∈ A' and y ∈ B'

⇒ y ∉ A and y ∉ B

⇒ y ∉ (A U B)

⇒ y ∈ (A U B)'

⇒ y ∈ P

Therefore, Q ⊂ P …………….. (ii)

Now combine (i) and (ii) we get; $P = Q$ i.e. $(A \cup B)' = A' \cap B'$

Proof of De Morgan's law 2:

$(A \cap B)' = A' \cup B'$

Let $M = (A \cap B)'$ and $N = A' \cup B'$

Let x be an arbitrary element of M then $x \in M \Rightarrow x \in (A \cap B)'$

$\Rightarrow x \notin (A \cap B)$

$\Rightarrow x \notin A$ or $x \notin B$

$\Rightarrow x \in A'$ or $x \in B'$

$\Rightarrow x \in A' \cup B'$

$\Rightarrow x \in N$

Therefore, $M \subset N$ …………….. (i)

Again, let y be an arbitrary element of N then $y \in N \Rightarrow y \in A' \cup B'$

$\Rightarrow y \in A'$ or $y \in B'$

$\Rightarrow y \notin A$ or $y \notin B$

$\Rightarrow y \notin (A \cap B)$

$\Rightarrow y \in (A \cap B)'$

$\Rightarrow y \in M$

Therefore, $N \subset M$ …………….. (ii)

Now combine (i) and (ii) we get; $M = N$ i.e. $(A \cap B)' = A' \cup B'$

Check Your Progress 2

1. Prove DeMorgan’s law with the help of example

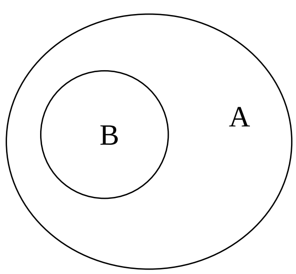
2. Explain the concept of symmetrical difference with the help of example

3. Define a. Complement of Set
   b. Disjoint Set

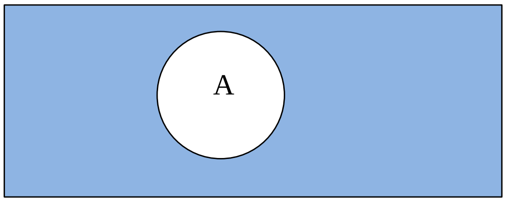
1.3 REPRESENTATION USING VENN DIAGRAM:

Venn diagram: Represents information that is easy in understanding

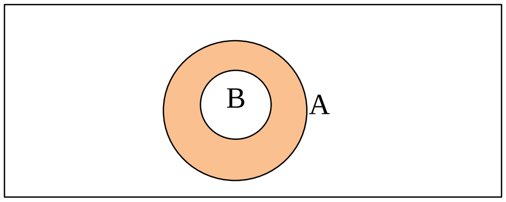
1. Set B is a proper subset of A



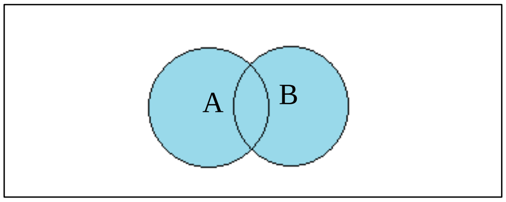
2. The absolute complement of set A



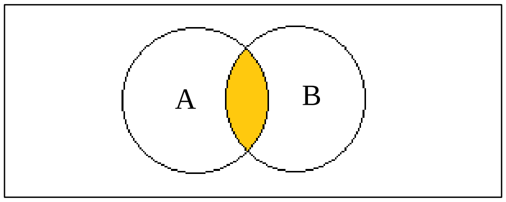
3. The relative complement of set B with respect to set A



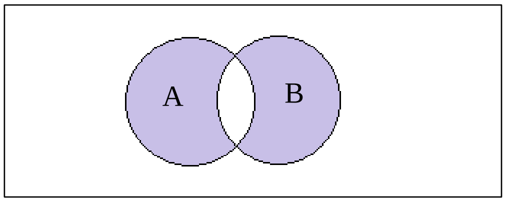
4. The Union of sets A and B



5. The intersection of sets A and B

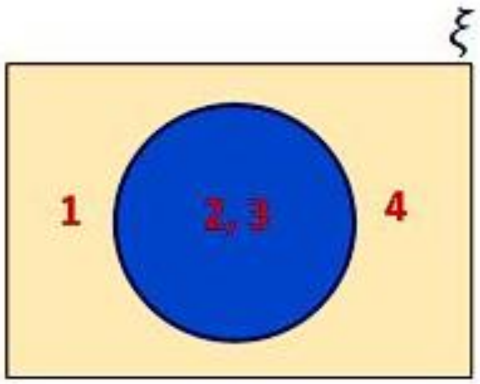


6. The symmetrical difference of sets A and B



Solved Example:

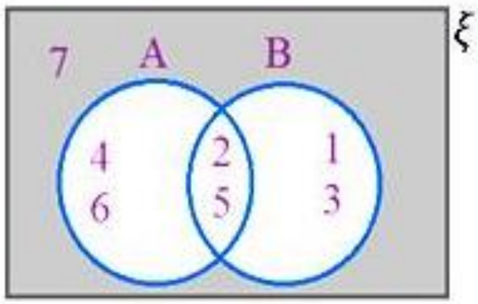
How to represent a set using Venn diagrams in different situations?



1. ξ is a universal set and A is a subset of the universal set.

ξ = {1, 2, 3, 4}

A = {2, 3}



2. **For example;**

Let ξ = {1, 2, 3, 4, 5, 6, 7}

A = {2, 4, 6, 5} and B = {1, 2, 3, 5}

Then A ∩ B = {2, 5}

1.4 TYPES OF SET

1.4.1 Cardinality Of A Set:

Let **‘S’** be a set. If there are exactly n distinct elements in **S**, where n is a nonnegative integer, we say S is a finite set and that n is the cardinality of **S**. The cardinality of **S** is denoted by | S |.

Examples:

1. V = {1,2,3,4,5}

   |V| = 5

2. A = {1,2,3,…, 20}

   |A| = 20

3. |∅| = 0

Note:

(i) Cardinal number of an infinite set is not defined.

(ii) Cardinal number of empty set is 0 because it has no element.

Example:

1. Write the cardinal number of each of the following sets:

(i) X = {letters in the word MALAYALAM}

(ii) Y = {5, 6, 6, 7, 11, 6, 13, 11, 8}

(iii) Z = {natural numbers between 20 and 50, which are divisible by 7}

Solution:

(i) Given, X = {letters in the word MALAYALAM}

Then, X = {M, A, L, Y}

Therefore, cardinal number of set X = 4, i.e., n(X) = 4

(ii) Given, Y = {5, 6, 6, 7, 11, 6, 13, 11, 8}

Then, Y = {5, 6, 7, 11, 13, 8}

Therefore, cardinal number of set Y = 6, i.e., n(Y) = 6

(iii) Given, Z = {natural numbers between 20 and 50, which are divisible by 7}

Then, Z = {21, 28, 35, 42, 49}

Therefore, cardinal number of set Z = 5, i.e., n(Z) = 5

1.4.2 Infinite Set:

A set is infinite if it is not finite.

Examples:

1. The set of natural numbers is an infinite set.
2. N = {1, 2, 3 ...}
3. The set of reals is an infinite set.

1.4.3 Power Set:

Given a set **S**, the power set of **S** is the set of all subsets of **S**. The power set is denoted by **P(S).**

Examples:

1. Assume an empty set ∅. What is the power set of ∅?

   P (∅) = {∅}

   What is the cardinality of P (∅)?

   | P (∅) | = 1.

2. Assume set {1} then P ({1}) = { ∅, {1} }

   |P ({1})| = 2

3. Assume {1,2,}

P({1,2,3}) = {∅, {1}, {2}, {3}, {1,2}, {1,3}, {2,3}, {1,2, 3}}

|P({1,2,3} | = 8

Computer Representation of Subsets of a Small Set S:

3- bit Binary Numbers			**S = {a, b, c}**
a	B	c	Elements of P(S)
0	0	0	∅
0	0	1	{c}
0	1	0	{b}

0	1	1	{b, c}
1	0	0	{a}
1	0	1	{a, c}
1	1	0	{a, b}
1	1	1	{a, b, c}

4. If S is a set with |S| = n then | P(S) | = 2^n

1.4.4 Product Of A Sets:

The Cartesian product of two sets A and B (also called the product set, set direct product, or cross product) is defined to be the set of all points (a, b) where a ∈ A and b ∈ B. It is denoted as A × B and is called the Cartesian product since it originated in Descartes' formulation of analytic geometry. In the Cartesian view, points in the plane are specified by their vertical and horizontal coordinates, with points on a line being specified by just one coordinate.

Examples:

S = {1, 2} and T = {a, b, c}

S × T = {(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)}

T x S = {(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)}

Note: S × T ≠ T × S.

Cardinality of the Cartesian product

|S × T| = |S| × |T|.

Example: Let A= {John, Peter, Mike} and B = {Jane, Ann, Laura}

A x B= {(John, Jane), (John, Ann), (John, Laura), (Peter, Jane), (Peter, Ann), (Peter, Laura), (Mike, Jane), (Mike, Ann), (Mike, Laura)}

|A x B| = 9

Also, $|A|=3$, $|B|=3 \longrightarrow |A|\ |B|= 9$

Example: If A = {7, 8} and B = {2, 4, 6}, find A × B.

Solution:

A × B = {(7, 2); (7, 4); (7, 6); (8, 2); (8, 4); (8, 6)}

The 6 ordered pairs thus formed can represent the position of points in a plane, if a and B are subsets of a set of real numbers.

Example: If A and B are two sets, and A × B consists of 6 elements: If three elements of A × Bare (2, 5) (3, 7) (4, 7) find A × B.

Solution:

Since, (2, 5) (3, 7) and (4, 7) are elements of A × B.

So, we can say that 2, 3, 4 are the elements of A and 5, 7 are the elements of B.

So, A = {2, 3, 4} and B = {5, 7}

Now, A × B = {(2, 5); (2, 7); (3, 5); (3, 7); (4, 5); (4, 7)}

Thus, A × B contain six ordered pairs.

1.4.5 Partition Of A Set:

A partition of a set ***X*** is a set of nonempty subsets of *X* such that every element ***x*** in **X** is in exactly one of these subsets (i.e., ***X*** is a disjoint of the subsets).

Equivalently, a family of sets *P* is a partition of ***X*** if and only if all of the following conditions hold:

1. ***P*** does not contain the empty set.
2. The union of the sets in ***P*** is equal to ***X***. (The sets in ***P*** are said to **cover *X***.)
3. The intersection of any two distinct sets in *P* is empty. (We say the elements of *P* are pairwise disjoint.)

In mathematical notation, these conditions can be represented as

1. $\emptyset \notin P$
2. $\bigcup_{A \in P} A = X$
3. If $A, B \in P$ and $A \neq B$ then $A \cap B = \emptyset$

where $\emptyset$ is the empty set.

The sets in *P* are called the blocks, parts or cells of the partition.

The **rank** of *P* is $|X| - |P|$, if *X* is finite.

1.4.6 Covering And Partition Of A Set:

Let **'S'** be a given set and $A = \{A_1, A_2, \ldots, A_m\}$where A_i, $i = 1, 2, \ldots, m$, is a subset of S and $\bigcup_{i=1}^{m} A_i = S$. Then the set A is called a covering of S. If in addition, the elements of **A**, which are subsets of S, are mutually disjoint, then A is called a partition of **S** and the sets $A_1, A_2, \ldots, A_m$ are called the blocks of the partition.

Example: Let S = {a, b, c} & consider the following collection of subset of S

A= {{a, b},{b, c}} B = {{a}, {a, c}} C = {{a},{b, c}} D = {{a, b, c}}

E = {{a}, {b}, {c}}F = {{a}, {a, b}, {a, c}}

The sets **A** and **F** are covering of **S** while **C, D, E** are partition of **S**. Of course every partition is also a covering. The set **B** is neither a covering nor a partition of **S**. The partition **D** has one block while **E** has three blocks. In fact, for any finite set, the small partition consists of the set itself as a block while the largest partition consists of blocks containing only single element. Two partitions are said to be equal as a sets. For a finite set, every partition is a finite partition i.e., every partition contains only a finite number of blocks.

Check Your Progress 3

1. Explain the concept of Cardinality of Set

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2. What do you understand by Partition of Set? Elaborate it.

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1.5 LET'S SUM UP

Set Theory is an ideal mathematical tool to understand and solve many problems and its operations and Venn diagrams are very useful to convert the difficult problems into simpler one to figure out the solution.

1.6 KEYWORDS

1. Set - A set is a collection of objects.
2. Element - An element or member of a set is an object that belongs to the set

3. Cardinality - The cardinality of a set is the number of distinct elements in the set
4. Universe of Discourse – the set containing all elements under discussion for a particular problem

1.7 QUESTIONS FOR REVIEW

1. If $A = \{1, 2, 3, 4\}$, $B = \{x\ : x$ is a positive integer and $x^2 < 18\}$.Is $A= B$?

2. Let $A = \{1, 2, 3\}$, $B = \{2, 3, 4, 5\}$. Find $A \Delta B$.

3. A software company requires 60 engineers to perform Java programming jobs and 35 engineers to perform c + + programming jobs. Out of this requirement, 15 are expected to perform both types of jobs. How many engineers have to be appointed for the purpose?

4. If A = { 1, 3, 5} and B = {2, 3}, then

Find: (i) A × B (ii) B × A (iii) A × A (iv) (B × B)

5. Each student in a class of 40 plays at least one indoor game chess, carom and scrabble. 18 play chess, 20 play scrabble and 27 play carom. 7 play chess and scrabble, 12 play scrabble and carom and 4 play chess, carom and scrabble. Find the number of students who play (i) chess and carrom. (ii) chess, carom but not scrabble

1.8 SUGGESTED READINGS

1. Kenneth H. Rosen - Discrete Mathematics and Its Applications, Tata Mc-Graw-Hill, 7th Edition, 2012.
2. Bernard Kolman, Robert C. Busby, Sharon Cutler Ross-Discrete Mathematical Structures-Prentice Hall, 3rd Edition, 1996.

3. Grimaldi R-Discrete and Combinatorial Mathematics. 1-Pearson, Addison Wesley, 5^{th} Edition, 2004.
4. C. L. Liu – Elements of Discrete Mathematics, McGraw-Hill, 1986.
5. F. Harary – Graph Theory, Addition Wesley Reading Mass, 1969.
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12. J. P. Tremblay & R. Manohar, Discrete Mathematical Structures with Applications to Computer Science, McGraw Hill Book Co. 1997
13. S. Witala, Discrete Mathematics - A Unified Approach, McGraw Hill Book Co

1.9 ANSWER TO CHECK YOUR PROGRESS

1. [HINT: Provide definition and different representation- 1.2]
2. [HINT: Provide definition – 1.2]
3. [HINT: Provide proof and example- 1.3.2]
4. [HINT: Provide the definition and example 1.3.1]
5. [HINT: Provide definition 1.3]
6. [HINT: Provide explanation – 1.5.1]
7. [HINT: Provide explanation 1.5.5]

UNIT 2: RELATIONS AND FUNCTIONS

STRUCTURE

2.0 OBJECTIVES

- Relations and Equivalence Relation
- Functions and its types

2.1 RELATIONS

First we will understand the ***Cartesian product*** which is a set of ***ordered pairs*** (two objects in specified order). Let A & B be the two sets and a ϵ A & b ϵ B. Thus, the ordered pair is (a, b). We can define the Cartesian Product of Set A & B

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

In mathematics there are various relations such as "is less than", "is parallel to", "is a power of ".

A ***relation*** from a set A to a set B is any subset of A×B (Cartesian Product of Set A & B) , is represented by 'R' which indicates relation such that **xRy** makes sense for **x ϵ A**and **y ϵ B.** We can represent R by the set of ordered pairs (x, y) for which the relation holds, that is **{(x, y) | xRy}.**

Example: If A= {1, 2, 3, 4} then the relation normally written x < y would be:

$$\{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$$

Domain of R: The domain of R is the set of first coordinates in R & denoted by **dom R**

From above example

dom R = {1,1,1,2,2,3} or { x |x ∈ A and (x, y) ∈ R for some y ϵ B}

Range of R: The range of R is the set of second coordinates in R & denoted by **ran R**

From above example

Ran R ={2, 3, 4, 3, 4, 4} or {y | y ϵ B and (x, y) ϵ R for some x ϵ A}

2.1.1 Equivalence Relation:

The three most important properties for a relation R on a set A are the reflexive, symmetric and transitive properties.

R is **reflexive** if xRx for all x.

R is **symmetric** if xRy→yRx for all x, y.

R is **transitive** if xRy and yRz→ xRz for all x, y, z.

Example:

1. Let A = {1, 2, 3, 4} & let R= {(1, 3), (4, 2), (2, 4), (2, 3), (3, 1)}

Not Reflexive: Since $(1, 1) \notin R$

Not Symmetric: $(2, 3) \in R$ but $(3, 2) \notin R$

Not Transitive: $(2, 3) \in R$ and $(3,1) \in R$ but $(2, 1) \notin R$

2. **$x R y \leftrightarrow |x - y| < 3$**

   Reflexive: Since for all x, $|x – x| = 0$ which is less than 3.

   Symmetric: Since $|y – x| = |x – y|$.

   Not Transitive: For example 1R3 and 3R5 but it is not true that 1R5.

2.1.2 Anti-symmetric Relation:

If following below condition is satisfied

$(a, b) \in R$ and $(b, a) \in R$ implies $a= b$

Refer the 1st example above a relation R on set A is not anti-symmetric as

$(1, 3) \in R$ and $(3, 1) \in R$ and $1 \neq 3$

2.1.3 Congruence Modulo:

Suppose **A** be the set of integers &**n** be fixed positive integer. So relation R_n on **A** by aR_nb if $a – b$ is an integral multiple of n. So $a – b =mn$ where **m** is some integer.

Thus **a** is congruent to **b** modulo **n**& is represented as **$a \equiv b \bmod n$**

Let's check the equivalence relation

Reflexive: $a – a = 0.n$ so that $a \equiv a \bmod n$ for each integer **n**

Symmetric: $a \equiv b \bmod n$ then $a – b = mn$ where **m** is some integer & $b – a = (–m) n$ so that we have $b \equiv a \bmod n$

Transitive: If $a \equiv b \bmod n$ & $b \equiv c \bmod n$ then $a – b = mn$ & $b – c = kn$ where **m**&**k** are integers. So after adding these two equations we have $a – c = (m + k) n$ & we can conclude that $a \equiv c \bmod n$.

2.1.4 Congruence Class:

If x is a given integer then [x] denote the set of all integers y such that **x ≡ y** mod n then [x] is known as Equivalence /Congruence Class containing **x** integer &**x**is called as representative of the congruence class.

[x] = {x + mn | m is any integer}

2.1.5 Composition of Relation:

- Let R be the relation from A to B & S a relation from B to C.
- The composition of R and S is denoted by **R · S / RS** is the relation from A to C which is given by **a RS c** if there is an element b ϵ B such that **a R b** and **b S c**
- Set B serves as an intermediary for establishing a correspondence between A & C.

Example:

1. If A ={1, 2, 3},B ={5,6} and C ={a, b, c}
   Let R = {(1,5), (1,6), (2,6)} be a relation from A to B &
   Let S = {(5, a), (6, c)} be a relation from B to C.
   Then RS= {(1,a), (1,c), (2, c)} is a relation from A to C

2. If R (x)= x + 1 and S (y) = y^2 are the functions defined on the set of real numbers then
   (R.S) (x) = S (R (x)) = $(x + 1)^2$ and
   (S.R) (x) = R (S(x)) = x^2+ 1

Example. In the given ordered pair (4, 6); (8, 4); (4, 4); (9, 11); (6, 3); (3, 0); (2, 3) find the following relations. Also, find the domain and range.

(a) Is two less than

(b) Is less than

(c) Is greater than

(d) Is equal to

Solution:

(a) R_1 is the set of all ordered pairs whose 1st component is two less than the 2nd component.

Therefore, $R_1 = \{(4, 6); (9, 11)\}$

Also, Domain (R_1) = Set of all first components of $R_1 = \{4, 9\}$ and
Range (R_2) = Set of all second components of $R_2 = \{6, 11\}$

(b) R_2 is the set of all ordered pairs whose 1st component is less than the second component.

Therefore, $R_2 = \{(4, 6); (9, 11); (2, 3)\}$.

Also, Domain $(R_2) = \{4, 9, 2\}$ and Range $(R_2) = \{6, 11, 3\}$

(c) R_3 is the set of all ordered pairs whose 1st component is greater than the second component.

Therefore, $R_3 = \{(8, 4); (6, 3); (3, 0)\}$

Also, Domain $(R_3) = \{8, 6, 3\}$ and Range $(R_3) = \{4, 3, 0\}$

(d) R_4 is the set of all ordered pairs whose 1st component is equal to the second component.

Therefore, $R_4 = \{(3, 3)\}$
Also, Domain (R) = $\{3\}$ and Range (R) = $\{3\}$

Example: Determine the domain and range of the relation R defined by

$R = \{x + 2, x + 3\} : x \in \{0, 1, 2, 3, 4, 5\}$

Solution:

Since, $x = \{0, 1, 2, 3, 4, 5\}$

Therefore,

$x = 0 \Rightarrow x + 2 = 0 + 2 = 2$ and $x + 3 = 0 + 3 = 3$

$x = 1 \Rightarrow x + 2 = 1 + 2 = 3$ and $x + 3 = 1 + 3 = 4$

$x = 2 \Rightarrow x + 2 = 2 + 2 = 4$ and $x + 3 = 2 + 3 = 5$

$x = 3 \Rightarrow x + 2 = 3 + 2 = 5$ and $x + 3 = 3 + 3 = 6$

$x = 4 \Rightarrow x + 2 = 4 + 2 = 6$ and $x + 3 = 4 + 3 = 7$

$x = 5 \Rightarrow x + 2 = 5 + 2 = 7$ and $x + 3 = 5 + 3 = 8$

Hence, $R = \{(2, 3), (3, 4), (4, 5), (5, 6), (6, 7), (7, 8)\}$
Therefore, Domain of $R = \{a : (a, b) \in R\}$ = Set of first components of all ordered pair belonging to R.

Therefore, Domain of $R = \{2, 3, 4, 5, 6, 7\}$

Range of $R = \{b : (a, b) \in R\}$ = Set of second components of all ordered pairs belonging to R.

Therefore, Range of $R = \{3, 4, 5, 6, 7, 8\}$

CHECK YOUR PROGRESS 1

1. Explain the concept of Equivalence Relations

2. Define

   a. Anti-Symmetry Relations

   b. Congruence Class

2.2 REPRESENTATION OF RELATION IN MATH:

The relation in math from set A to set B is expressed in different forms.

(i) Roster form

(ii) Set builder form

(iii) Arrow diagram

2.3.1 Roaster Form:

- In this, the relation (R) from set A to B is represented as a set of ordered pairs.
- In each ordered pair 1st component is from A; 2nd component is from B.
- Keep in mind the relation we are dealing with. (>, < etc.)

For Example:

1. If A = {p, q, r} B = {3, 4, 5}

then R = {(p, 3), (q, 4), (r, 5)}

Hence, R ⊆ A × B

2. Given A = {3, 4, 7, 10} B = {5, 2, 8, 1} then the relation R from A to B is defined as 'is less than' and can be represented in the roster form as R = {(3, 5) (3, 8) (4, 5), (4, 8), (7, 8)}

Here, 1^{st} component < 2^{nd} component.

In roster form, the relation is represented by the set of all ordered pairs belonging to R.

If A = {-1, 1, 2} and B = {1, 4, 9, 10}

if a R b means $a^2 = b$

then, R (in roster form) = {(-1, 1), (1, 1), (2, 4)

2.3.2 Set Builder Form:

In this form, the relation R from set A to set B is represented as R = {(a, b): a ∈ A, b ∈ B, a...b}, the blank space is replaced by the rule which associates a and b.

For Example:

Let A = {2, 4, 5, 6, 8} and B = {4, 6, 8, 9}

Let R = {(2, 4), (4, 6), (6, 8), (8, 10) then R in the set builder form, it can be written as

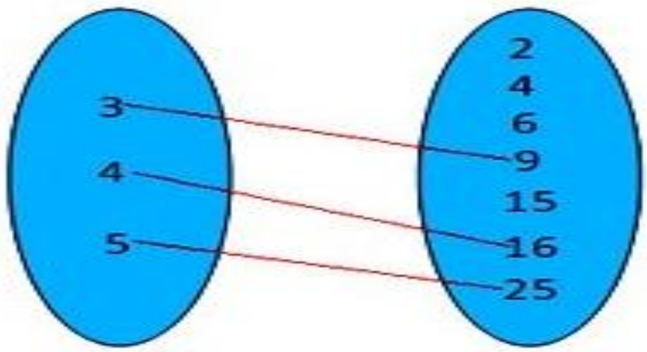
R = {a, b} : a ∈ A, b ∈ B, *a* is 2 less than *b*}

2.3.3 Arrow Diagram:

Steps:

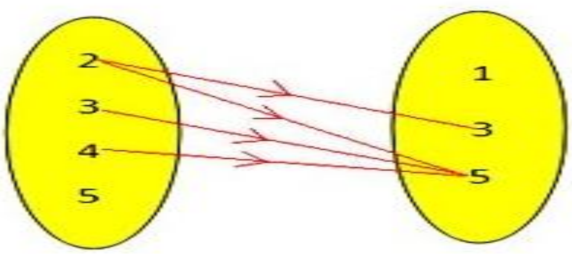
- ✓ Draw two circles representing Set A and Set B.
- ✓ Write their elements in the corresponding sets, i.e., elements of Set A in circle A and elements of Set B in circle B.
- ✓ Draw arrows from A to B which satisfy the relation and indicate the ordered pairs.

Example: If A = {3, 4, 5} B = {2, 4, 6, 9, 15, 16, 25}, then relation R from A to B is defined as 'is a positive square root of' and can be represented by the arrow diagram as shown.
Here R = {(3, 9); (4, 16); (5, 25)}



In this form, the relation R from set A to set B is represented by drawing arrows from 1st component to 2nd components of all ordered pairs which belong to R.

Example: If A = {2, 3, 4, 5} and B = {1, 3, 5} and R be the relation 'is less than' from A to B,then R = {(2, 3), (2, 5), (3, 5), (4, 5)}



Example: Let A = {2, 3, 4, 5} and B = {8, 9, 10, 11}.

Notes

Let R be the relation 'is factor of' from A to B.

(a) Write R in the roster form. Also, find Domain and Range of R.

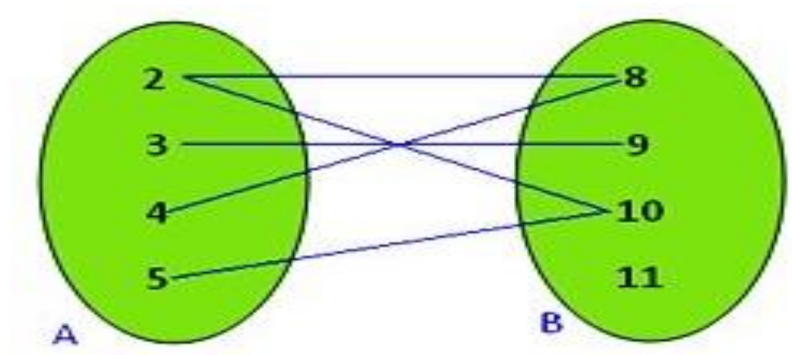
(b) Draw an arrow diagram to represent the relation

Solution: (a) Clearly, R consists of elements (a, b) where a is a factor of b.

Therefore, Relation (R) in the roster form is R = {(2, 8); (2, 10); (3, 9); (4, 8), (5, 10)}

Therefore, Domain (R) = Set of all first components of R = {2, 3, 4, 5} and Range (R) = Set of all second components of R = {8, 10, 9}

b) The arrow diagram representing R is as follows



Check Your Progress 2

1. Explain different types of representation used for relations. Give examples

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2. Highlight the steps to draw the arrow diagram

2.3 FUNCTIONS

A function 'f' from a set **A** to a set **B** is the pair of sets (A, B) with a rule that associates with each element '**x -- a unique element of B'**, such that $x \epsilon A$ and written as$f(x)$. This element is called the ***image*** of f. We indicate that f is a function from A to B by writing f: A →B. The set A is called the ***domain*** of f and is called the ***codomain.*** The synonyms of "function" are "mapping", "transformation", "correspondence" and operator. Other Way to understand ***'Function'***: Let A & B be the two non-empty sets. Now a function f: A→ B is a relation from A to B such that:

- Dom f = A, for each a ϵ A, (a, b) ∈f for some b ϵ B which refers that f is defined at each a ∈ A.
- If (a,b) ∈f and (a, c) ∈f then b = c. In this, we can refer f is well defined or single valued. Thus no element of A is related to two elements of B
- If (a, b)∈f then b is known as the image of a under f& we can represent it in the following way

  $b = f(a)$

2.4.1 One-To-One Function:

If a function says$f$: A→ B satisfy the following condition below:

- $f(x\mathbf{1}) = y$ and $f(x\mathbf{2}) = y \xrightarrow{\text{implies}} x\mathbf{1} = x\mathbf{2}$
- For each b ∈ ran$f$, f^{-1}(b) contains only one element.
- Let f^{-1}(b) as the set of preimages of b, for each b ϵ ran f, b has precisely one preimage

2.4.2 Onto Function:

If a function says f = A→ B satisfies the following condition below:

- ran f= B or
- f^{-1}(b) is non empty for each b ϵ B
- For each b ϵ B has some preimage in A

2.4.3 Inverse Function:

Let a function f : A→ B is one-to-one and onto function, then the inverse relation $f^{-1}$is single valued, & thus is a function from B to A. In this case $f^{-1}$is the inverse function of f.

Note:

- ✓ A one to one & onto function f : A→ B is also known as one-to-one correspondence between A & B
- ✓ If a function$f$: A→ B is not one-to-one then it is called as ***many-to-one function***. Also if f not onto B then it is known as ***into*** B
- ✓ Two functions are equal if they are equal as a set because function is also a set

Examples:

1. Let A = {r, s, t}, B = {1, 2, 3} and C ={r, s, t, u}. So R= {(r, 1), (r, 2), (t, 2)} is a relation from A to B but R is not a function since R (r) ={1,2}
2. The set f = {(r, 1), (s, 2),(t, 2)} is a function from A to B but f is not one-to-one since
   f^{-1}(2) = {s, t}& also f is not onto B as f^{-1}(3)= ϕ
3. The function **g = {(r, 1), (s, 2), (t, 3)}** is both one-to-one & onto function from A to B**.**
   Also g^{-1} = {(1, r), (2, s), (3, t)}
4. The function **h: C → B** defined as **h= {(r, 1), (s, 1), (t, 2), (u, 3**) is onto but not one-to-one function since **h^{-1}(1) = {r, s}**

2.4.4 Composition Of A Function:

Let f be a function from set A to set B and let g be a function from set B to set C. The composition of the functions g and f, denoted by g ·f is defined by

(g ·f)(a) = g (f (a))

Example:

1. Let A = {1, 2, 3} and B = {a, b, c, d}.

   Then ***g***: A ⟶ A   ***f***: A⟶B

   1⟶3   1⟶ b

   2⟶1   2⟶a

   3⟶2   3⟶d

   So, the composition of function is given as ***f*** ·***g***: A⟶B

   1⟶d

   2⟶b

   3⟶a

2. $(f \cdot f)(x)\ and\ (f^{-1} \cdot f)(x) = x, for\ all\ x.$

   Let f: R⟶R, where *f*(*x*) = 2*x* – 1 and $f^{-1}(x) = {(x+1)}/{2}$

   $(f^{-1} \cdot f)(x) = f\ [f^{-1}(x)]$

   $= f[\,{(x+1)}/{2}]$

   $= 2\left[\frac{x+1}{2}\right] - 1$

   $= (x+1) - 1$

$$= x$$

2.4.5 Hash Function:

Suppose we wish to retrieve some information stored in a table of size n with indexes 0, 1 . . . n – 1. The items in the table can be very general things. For example, the items might be strings of letters, or they might be large records with many fields of information. To look up a table item we need a key to the information we desire.

For example, if the table contains records of information for the 12 months of the year, the keys might be the three-letter abbreviations for the 12 months. To look up the record for January, we would present the key Jan to a lookup program. The program uses the key to find the table entry for the January record of information. Then the information would be available to us. An easy way to look up the January record is to search the table until the key Jan is found. This might be OK for a small table with 12 entries. But it may be impossibly slow for large tables with thousands of entries. Here is the general problem that we want to solve. Given a key, find the table entry containing the key without searching. This may seem impossible at first glance. But let's consider away to use a function to map each key directly to its table location.

We can define hash function is a function that maps a set S of keys toa finite set of table indexes, which we'll assume are 0, 1 ... n – 1. A table whose information found by a hash function called a hash table.

For example, let **S** be the set of three-letter abbreviations for the months of the year. We might define a hash function ***f***: S → {0, 1 ... 11} in the following way.

$$f\,(XYZ) = (ord\,(X) + ord\,(Y) + ord\,(Z)) \bmod 12.$$

Where **or** (**X**) denotes the integer value of the ASCII code for **X**.(The ASCII values for A to Z and **a** to **z** are 65 to 90 and 97 to 122, respectively.)

For example, we'll compute the value for the key Jan.

f(JAN) = [**ord**(**X**) + **ord**(**Y**) + **ord**(n)]**mod** 12 =(74+97+110) **mod** 12 = 5.

2.4.6 Characteristics Function:

Consider some universal set 'U'. Let A ⊆ U. The function $\chi_A : U \rightarrow \{0,1\} defined\ by$

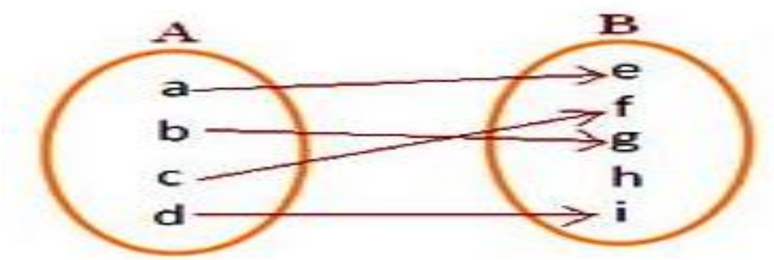
$\chi_A(x) = 1, if\ x\ \in A$

$\chi_A(x) = 0, if\ x\ \in A^c$ is called the characteristic function of A.

2.5 REPRESENTATION OF A FUNCTION BY AN ARROW DIAGRAM:

In this, we represent the sets by closed figures and the elements are represented by points in the closed figure.

The mapping f : A → B is represented by arrow which originates from elements of A and terminates at the elements of B.



<u>Example:</u> Let A = {1, 2, 3, 4} and B = {0, 3, 6, 8, 12, 15}

Consider a rule f (x) = x^2 - 1, x∈A, then

(a) show that f is a mapping from A to B

(b) draw the arrow diagram to represent the mapping.

(c) represent the mapping in the roster form.

(d) write the domain and range of the mapping

Solution:

a) Using f (x) = $x^2 - 1$, $x \in A$ we have

f(1) = 0,

f(2) = 3,

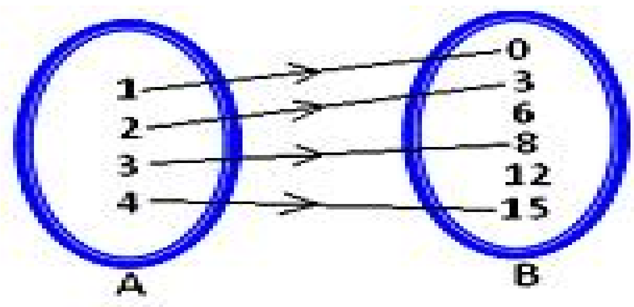
f(3) = 8,

f(4) = 15

We observe that every element in set A has unique image in set B.

Therefore, f is a mapping from A to B.

(b) Arrow diagram which represents the mapping is given below



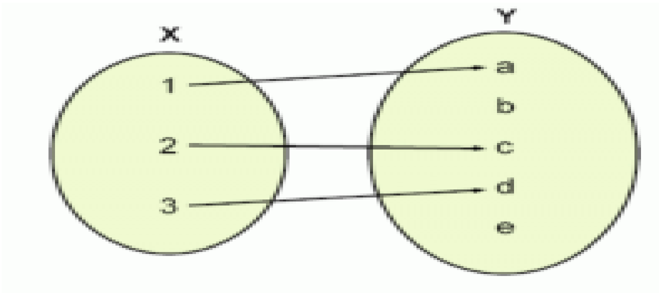
(c) Mapping can be represented in the roster form as

f = {(1, 0); (2, 3); (3, 8); (4, 15)}

(d) Domain (f) = {1, 2, 3, 4} Range (f) = {0, 3, 8, 15}

Notes

Example: The given function is:



a) Injective b) Surjective c) Many to one d) Bijective

Ans. a) Injective each element in X is mapped to a distinct element in Y.

CHECK YOUR PROGRESS 3

1. Define Inverse Function

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2. What do you understand by the Hash Function?

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2.4 LET'S SUM UP

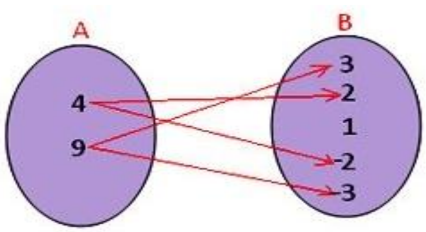
Relations may exist between objects of the same set or between objects of two or more sets. It helps us to establish relationship between elements of the set. There are many business concepts, science concepts where relations and functions are used to establish or verify different facts.

2.5 KEYWORDS

1. Relation: **The relation** is the subset of the Cartesian product which contains only some of the ordered pair based on the relationships defined between the first and second elements.
2. Functions: If every element of a set A is related with one and only one element of another set then this kind of relation qualifies as a **function**.
3. Domain: is the set of all first elements of R.
4. Range: is the set of all second elements of R.

2.6 QUESTIONS FOR REVIEW

1. The adjoining figure shows a relation between the sets A and B.



Write this relation in

• Set builder form

• Roster form

• Find the domain and range

2.Let A = {2, 3, 4, 5} and B = {8, 9, 10, 11}.

Let R be the relation ‘is factor of’ from A to B.

(a) Write R in the roster form. Also, find Domain and Range of R.

(b) Draw an arrow diagram to represent the relation

3.Let T be a set of triangles in a plane, and define R as the set R = {(a, b); a, $b \in T$, a is congruent to b}. Is R is an equivalence relation?

4.Show that the relation of congruence modulo m, defined on the set **Z** of integers by $a \equiv b \pmod{m}$ is an equivalence relation.

5. Function f is defined by $f(x) = -2x^2 + 6x - 3$, find f(- 2).

2.7 SUGGESTED READINGS

- Kenneth H. Rosen - Discrete Mathematics and Its Applications, Tata Mc-Graw-Hill, 7th Edition, 2012.
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- J. P. Tremblay & R. Manohar, Discrete Mathematical Structures with Applications to Computer Science, McGraw Hill Book Co. 1997
- S. Witala, Discrete Mathematics - A Unified Approach, McGraw Hill Book Co

2.8 ANSWER TO CHECK YOUR PROGRESS

1. [HINT: Provide the explanation 2.2.1]
2. [HINT: Provide the definition – 2.2.2 & 2.2.4]
3. [HINT: Provide the different types of representation -2.3]
4. [HINT: Provide the steps – 2.3.3]
5. [HINT: Provide definition – 2.4.3]
6. [HINT: Provide definition – 2.4.5]

UNIT 3: BOOLEAN ALGEBRA

STRUCTURE

3.0 OBJECTIVE

- ✓ Learn the concept of Boolean Algebra
- ✓ Understand Boolean Homomorphism and Isomorphism
- ✓ Comprehend the concept of induced partial order
- ✓ What is atom?

3.1 INTRODUCTION

Boolean logic is an abstract mathematical structure named after the famous Mathematician George Boole. Boole tried to formalize the process of logical reasoning using symbols instead of words. Boolean Algebra provides us a basic logic for the operations on binary numbers 0, 1.

3. 2 DEFINITION

Notes

A **Boolean algebra** is a nonempty set S which is closed under the binary operations ∨ (called **join**), ∧ (called **meet**), and the unary operation ¬ (called **inverse** or **complement**) satisfying the following properties for all x, y, z ∈ S:

1. [Commutativity] : $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$.

2. **[Distributivity]** : $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

3. **[Identity elements]** : There exist elements **0**, **1** ∈ S such that $x \vee \mathbf{0} = x$ and $x \wedge \mathbf{1} = x$.

4. **[Inverse]** : $x \vee \neg x = \mathbf{1}$ and $x \wedge \neg x = \mathbf{0}$.

When required, we write the Boolean algebra S as (S, ∨, ∧, ¬) showing the operations explicitly.
Notice that the fourth property in the definition above uses the two special elements **0** and **1**,
whose existence has been asserted in the third property. This is meaningful when these two elements are uniquely determined by the third property.

Note:
(i) a' is called the complement of a. (a')' will be denoted by a'' and so on. Very often we shall write a · b as ab.

(ii) The binary operations in the definition need not be written as + and

Instead, we may use other symbols such as ∪, ∩ (known as union and intersection respectively), or, ⋁, ⋀ (known as join and meet) to denote these operations.

(iii) A Boolean algebra is generally denoted by a 6-tuple (B, +, ·, ', 0, 1) or by (B, +, ·, ') or, simply by the set B in it.

Examples:

1. Let A be a non-empty set and P(A) be the power set of A. Then P(A) is a Boolean algebra under the usual operations of union, intersection and complementation in P(A). The sets ∅ and A are the zero element and unit element of the Boolean algebra P(A). Observe that if A is an infinite set, then the Boolean algebra P(A) will contain infinite number of elements.

2. Let B be the set of all positive integers which are divisors of 70; i.e., B = {1, 2, 5, 7, 10, 14, 35, 70}. For any a, b ϵ B, let a + b = l.c.m of a, b; a · b = h.c.f. of a, b and a' = ^{7}<span style='font-size: 50%'>/$_{0}$. Then with the help of elementary properties of l.c.m. and h.c.f. it can be easily verified that (B, +, ·, ', 1, 70) is a Boolean algebra. Here 1 is the zero element and 70 is the unit element.

Proposition. Let S be a Boolean algebra. Then the following statements are true:

1. Elements **0** and **1** are unique.
2. Corresponding to each s ∈ S, ¬s is the unique element in S that satisfies the property: s∨¬s = **1**
and s ∧ ¬s = **0**.
3. For each s ∈ S, ¬¬s = s.

Proof. (1) Let **0**1, **0**2 ∈ S be such that for each x ∈ S, x ∨ **0**1 = x and x ∨ **0**2 = x. Then, in particular,

02 ∨ **0**1 = **0**2 and **0**1 ∨ **0**2 = **0**1.

By Commutatively, **0**2 ∨ **0**1 = **0**1 ∨ **0**2. So, **0**2 = **0**1.

That is, **0** is the unique element satisfying the property that for each $x \in S$, $\mathbf{0} \vee x = x$. A similar argument shows that **1** is the unique element that satisfies the property that for each $x \in S$, $x \wedge \mathbf{1} = x$.

(2) Let $s \in S$. By definition, $\neg s$ satisfies the required properties.

For the converse, suppose $t, r \in S$ are such that $s \vee t = \mathbf{1}$, $s \wedge t = \mathbf{0}$, $s \vee r = \mathbf{1}$ and $s \wedge r = \mathbf{0}$.

Then

$t = t \wedge \mathbf{1} = t \wedge (s \vee r) = (t \wedge s) \vee (t \wedge r) = \mathbf{0} \vee (t \wedge r) = (s \wedge r) \vee (t \wedge r) = (s \vee t) \wedge r = \mathbf{1} \wedge r = r.$

(3) It directly follows from the definition of inverse, due to commutatively.

Example :

1. Let S be a nonempty set. Then P(S) is a Boolean algebra with $\vee = \cup$, $\wedge = \cap$, $\neg A = A^c$, $\mathbf{0} = \emptyset$
and $\mathbf{1} = S$. This is called the **power set Boolean algebra**. So, we have Boolean algebras of
finite size as well as of uncountable size.

2. Take $D(30) = \{n \in N : n \mid 30\}$ with $a \vee b = \mathrm{lcm}(a, b)$, $a \wedge b = \gcd(a, b)$ and $\neg a = 30\ a$. It is a
Boolean algebra with $\mathbf{0} = 1$ and $\mathbf{1} = 30$.

3. Let $B = \{T, F\}$, where $\vee$, $\wedge$ and $\neg$ are the usual connectives. It is a Boolean algebra with $\mathbf{0} = F$
and $\mathbf{1} = T$.

4. Let B be the set of all truth functions involving the variables p1, . . . , pn, with usual operations

$\vee$, $\wedge$ and $\neg$. Then B is a Boolean algebra with $\mathbf{0} = \perp$ and $\mathbf{1} = >$. This is called the **free Boolean algebra** on the generators $p_1, \ldots, p_n$.

5. The set of all formulas (of finite length) involving variables p1, p2, . . . is a Boolean algebra with usual operations. This is also called the free Boolean algebra on the generators p1, p2, ... Here also $\mathbf{0} = \perp$ and $\mathbf{1} = T$. So, we have a Boolean algebra of denumerable size.

Remark : The rules of Boolean algebra treat ($\vee$, $\mathbf{0}$) and ($\wedge$, $\mathbf{1}$) equally. Notice that the second

parts in the defining conditions of above definition of Boolean Algebra can be obtained from the corresponding first parts by replacing $\vee$ with $\wedge$, $\wedge$ with $\vee$, $\mathbf{0}$ with $\mathbf{1}$, and $\mathbf{1}$ with $\mathbf{0}$ simultaneously. Thus, any statement that one can derive from these assumptions has a dual version which is derivable from the same assumptions. This is called the **principle of duality**.

Theorem [Laws] Let S be a Boolean algebra. Then the following laws hold for all $s, t \in S$:

1. **[Constants]** : $\neg\mathbf{0} = \mathbf{1}$, $\neg\mathbf{1} = \mathbf{0}$, $s \vee \mathbf{1} = \mathbf{1}$, $s \wedge \mathbf{1} = s$, $s \vee \mathbf{0} = s$, $s \wedge \mathbf{0} = \mathbf{0}$.
2. **[Idempotence]** : $s \vee s = s$, $s \wedge s = s$.
3. **[Absorption]** : $s \vee (s \wedge t) = s$, $s \wedge (s \vee t) = s$.
4. **[Cancellation]** : $s \vee t = r \vee t$, $s \vee \neg t = r \vee \neg t \Rightarrow s = r$.
5. **[Cancellation]** : $s \wedge t = r \wedge t$, $s \wedge \neg t = r \wedge \neg t \Rightarrow s = r$.
6. **[Associativity]** : $(s \vee t) \vee r = s \vee (t \vee r)$, $(s \wedge t) \wedge r = s \wedge (t \wedge r)$.

Proof. We give the proof of the first part of each item and that of its dual is left for the reader.

(1) $\mathbf{1} = \mathbf{0} \vee (\neg\mathbf{0}) = \neg\mathbf{0}$.

$s \vee \mathbf{1} = (s \vee \mathbf{1}) \wedge \mathbf{1} = (s \vee \mathbf{1}) \wedge (s \vee \neg s) = s \vee (\mathbf{1} \wedge \neg s) = s \vee \neg s = \mathbf{1}$.

$s \vee \mathbf{0} = s \vee (s \wedge \neg s) = (s \vee s) \wedge (s \vee \neg s) = s \wedge \mathbf{1} = s$.

(2) $s = s \vee \mathbf{0} = s \vee (s \wedge \neg s) = (s \vee s) \wedge (s \vee \neg s) = (s \vee s) \wedge \mathbf{1} = (s \vee s)$.

Notes

(3) $s \vee (s \wedge t) = (s \wedge \mathbf{1}) \vee (s \wedge t) = s \wedge (\mathbf{1} \vee t) = s \wedge \mathbf{1} = s$.

(4) Suppose that $s \vee t = r \vee t$ and $s \vee \neg t = r \vee \neg t$.

Then

$s = s \vee \mathbf{0} = s \vee (t \wedge \neg t) = (s \vee t) \wedge (s \vee \neg t) = (r \vee t) \wedge (r \vee \neg t) = r \vee (t \wedge \neg t)$

$= r \vee \mathbf{0} = r$.

(5) This is the dual of (4) and left as an exercise.

(6) Using distributivity and absorption, we have

$s \vee (t \vee r) \wedge \neg s = (s \wedge \neg s) \vee (t \vee r) \wedge \neg s = \mathbf{0} \vee (t \vee r) \wedge \neg s$

$= (t \vee r) \wedge \neg s = (t \wedge \neg s) \vee (r \wedge \neg s)$.

$(s \vee t) \vee r \wedge \neg s = (s \vee t) \wedge \neg s \vee (r \wedge \neg s) = (s \wedge \neg s) \vee (t \wedge \neg s) \vee (r \wedge \neg s)$

$= (\mathbf{0} \vee (t \wedge \neg s) \vee (r \wedge \neg s) = (t \wedge \neg s) \vee (r \wedge \neg s)$.

Hence, $s \vee (t \vee r) \wedge \neg s = (s \vee t) \vee r \wedge \neg s$.

Also, $(s \vee t) \vee r \wedge s = (s \vee t) \wedge s \vee (r \wedge s) = s \vee (r \wedge s) = s = s \vee (t \vee r) \wedge$
s.

Now, apply Cancellation law to obtain the required result.

Isomorphisms between two similar algebraic structures help us in understanding an unfamiliar entity through a familiar one. Boolean algebras are no exceptions.

Check Your Progress 1

1. Explain the terms
   a. Boolean Algebra
   b. Power set Boolean Algebra

2. State and prove the Associativity law

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3. 3 BOOLEAN HOMOMORPHISM AND ISOMORPHISM

CONCEPT: Let (B1, ∨ 1, ∧ 1, ¬1) and (B2, ∨ 2, ∧ 2, ¬2) be two Boolean algebras. A function f : B1 → B2 is a **Boolean homomorphism** if it preserves **0**, **1**, ∨ , ∧ , and ¬. In such a case, f(**0**1) = **0**2, f(**1**1) = **1**2, f(a ∨ 1 b) = f(a) ∨ 2 f(b), f(a ∧ 1 b) = f(a) ∧ 2 f(b), f(¬1a) = ¬2f(a).

CONCEPT: A **Boolean isomorphism** is a Boolean homomorphism which is a bisections.

Unless we expect an ambiguity in reading and interpreting the symbols, we will not write the

subscripts with the operations explicitly as is done in above Definition.

Example. Recall the notation [n] = {1, 2, . . . , n}. The function f : P([4]) → P([3]) defined

by f(S) = S \ {4} is a Boolean homomorphism.

We check two of the properties

f(A ∨ B) = f(A ∪ B) = (A ∪ B) \ {4} = (A \ {4}) ∪ (B \ {4}) = f(A) ∨ f(B).

f(**1**) = f([4]) = [4] \ {4} = [3] = **1**.

Let (L, ≤) be a distributive complemented lattice. Then, L has two binary operations ∨ and ∧ and the unary operation ¬x. It can be verified that (L, ∨ ,

$\wedge$, $\neg$) is a Boolean algebra. Conversely, let (B, $\vee$, $\wedge$, $\neg$) be a Boolean algebra. Is it possible to define a partial order $\leq$ on L so that (B, $\leq$) will be a distributive complemented lattice, and then in this lattice, the resulting operations of $\vee$, $\wedge$ and $\neg$ will be the same operations we have started with?

Theorem : Let (B, $\vee$, $\wedge$, $\neg$) be a Boolean algebra. Define the relation $\leq$ on B by

$a \leq b$ if and only if $a \wedge b = a$ for all $a, b \in B$.

Then (B, $\leq$) is a distributive complemented lattice in which lub$\{a, b\} = a \vee b$ and glb$\{a, b\} = a \wedge b$ for all $a, b \in B$.

Proof. We first verify that (B, $\leq$) is a partial order.

Reflexive: $s \leq s$ if and only if $s \wedge s = s$, which is true.

Ant symmetry: Let $s \leq t$ and $t \leq s$. Then we have $s = s \wedge t = t$.

Transitive: Let $s \leq t$ and $t \leq r$. Then $s \wedge t = s$ and $t \wedge r = t$.

Using associativity,

$s \wedge r = (s \wedge t) \wedge r = s \wedge (t \wedge r) = s \wedge t = s$; consequently, $s \leq r$.

Now, we show that $a \vee b$ = lub$\{a, b\}$.

Since B is a Boolean algebra, using absorption, we get $(a \vee b) \wedge a = a$ and hence $a \leq a \vee b$.

Similarly, $b \leq a \vee b$. So, $a \vee b$ is an upper bound for $\{a, b\}$.

Now, let x be any upper bound for $\{a, b\}$.

Then, by distributive property,

$(a \vee b) \wedge x = (a \wedge x) \vee (b \wedge x) = a \vee b$. So, $a \vee b \leq x$.

Thus, $a \vee b$ is the lub of $\{a, b\}$.

Analogous arguments show that

a ∧ b = glb{a, b}.

Since for all a, b ∈ B, a∨ b and a∧ b are in B, we see that lub{a, b} and glb{a, b} exist. Thus (B, ≤) is a lattice.

Further, if a ∈ B, then ¬a ∈ B. This provides the complement of a in the lattice (B, ≤). Further, both the distributive properties are already satisfied in B. Hence (B, ≤) is a distributive complemented lattice.

In view of above Theorem, we give the following definition.

3.4 INDUCED PARTIAL ORDER

DEFINITION: Let (B, ∨ , ∧ , ¬) be a Boolean algebra. The relation ≤ on B given by a ≤ b if and only if a ∧ b = a for all a, b ∈ Bis called the **induced partial order**.

A minimal element of B with respect to the partial order ≤, which is different from **0** is called an **atom** in B.

EXAMPLE:

1. In the power set Boolean algebra, singleton sets are the only atoms.
2. The {F, T } Boolean algebra has only one atom, namely T .

Proposition 3.3.1. Each finite Boolean algebra has at least one atom.

Proof. Let B be a finite Boolean algebra. Assume that no element of B is an atom. Now, **0** < **1** and **1** is not an atom. Then there exists b1 ∈ B such that $0 < b_1 <$ **1**. Since b1 is not an atom, there exists $b_2 \in B$ such that $0 < b_2 < b_1 <$ **1**. By induction it follows that we have a sequence of elements (b_i) such that $0 < \cdots < b_i < b_{i-1} < \cdots < b_1 <$ **1**. As B is finite, there exist k > j such that $b_k = b_j$. We

then have $b_k < b_{k-1} < \cdot \cdot \cdot < b_j = b_k$. This is impossible. Hence B has at least one atom.

Proposition 3.3.2. Let p and q be atoms in a Boolean algebra B. If $p \neq q$, then $p \wedge q =$ **0**.

Proof. Suppose that $p \wedge q$ 6= **0**. We know that $p \wedge q \leq p$. If $p \wedge q \neq p$, then $p \wedge q < p$. But this is not

possible since p is an atom. So, $p \wedge q = p$. Similarly, $q \wedge p = q$. By commutativity, $p = p \wedge q = q \wedge p = q$.

Theorem 3.4.1 [Representation]

Let B be a finite Boolean algebra. Then there exists a set X

such that B is isomorphic to P(X).

Proof. Let X be the set of all atoms of B. By Proposition 8.3.13, $X \neq \emptyset$.

Define $f : B \rightarrow P(X)$ by $f(b) = \{x \in B : x$ is an atom and $x \leq b\}$ for $b \in B$.

We show that f is the required Boolean isomorphism.

Injection: Suppose $b_1 \neq b_2$. Then, either $b_1 \nleq b_2$ or $b_2 \nleq b_1$. Without loss of generality, let b1 b_2.

Note that $b_1 = b_1 \wedge (b_2 \vee \neg b_2) = (b_1 \wedge b_2) \vee (b1 \wedge \neg b2)$. Also, the assumption b1 b2 implies $b1 \wedge b_2 \neq b_1$ and hence $b_1 \wedge \neg b2 \neq$ **0** .So, there exists an atom $x \leq (b1 \wedge \neg b2)$ and hence $x = x \wedge b_1 \wedge \neg b_2$.

Then $x \wedge b_1 = (x \wedge b_1 \wedge \neg b_2) \wedge b_1 = x \wedge b_1 \wedge \neg b_2 = x$.

Thus, $x \leq b_1$. Similarly, $x \leq \neg b_2$. As $x \neq$ **0**, we cannot have $x \leq b_2$ (for, $x \leq \neg b_2$ and $x \leq b_2$ imply

$x \leq b_2 \wedge \neg b_2 =$ **0**). Thus there is an atom in f(b1) which is not in $f(b_2)$.

Therefore, $f(b_1) \neq f(b_2)$.

Surjection: Let $A = \{x1, \ldots, xk\} \subseteq X$. Write $a = x1 \vee \cdots \vee xk$ (if $A = \emptyset$, take $a = \mathbf{0}$). Clearly,

$A \subseteq f(a)$. We show that $A = f(a)$.

So, let $y \in f(a)$.

Then y is an atom in B and $y = y \wedge a = y \wedge (x1 \vee \cdots \vee xk) = (y \wedge x1) \vee \cdots \vee (y \wedge xk)$.

Since $y \neq \mathbf{0}$, by Proposition 3.3.2, $y \wedge xi \neq \mathbf{0}$ for some $i \in \{1, 2, \ldots, k\}$. As x_i and y are atoms, we have $y = y \wedge x_i = x_i$ and hence $y \in A$. That is, $f(a) \subseteq A$ so that $f(a) = A$.

Thus, f is a surjection.

Preserving $\mathbf{0}$, $\mathbf{1}$: Clearly $f(\mathbf{0}) = \emptyset$ and $f(\mathbf{1}) = X$.

Preserving $\vee$, $\wedge$: By definition,

$x \in f(b_1 \wedge b_2) \Leftrightarrow x \leq b_1 \wedge b_2 \Leftrightarrow x \leq b_1$ and $x \leq b_2$

$\Leftrightarrow x \in f(b_1)$ and $x \in f(b_2) \Leftrightarrow x \in f(b_1) \cap f(b_2)$.

For the other one, let $x \in f(b_1 \vee b_2)$. Then, $x = x \wedge (b_1 \vee b_2) = (x \wedge b_1) \vee (x \wedge b_2)$. So, $x \wedge b_1 \neq \mathbf{0}$ or $x \wedge b_2 \neq \mathbf{0}$.

Without loss of generality, suppose $x \wedge b_1 \neq \mathbf{0}$. As x is an atom, $x \leq b_1$ and hence $x \in f(b_1) \subseteq f(b_1) \cup f(b_2)$. Conversely, let $x \in f(b_1) \cup f(b_2)$.

Without loss of generality, let $x \in f(b_1)$. Thus, $x \leq b_1$ and hence $x \leq b_1 \vee b2$ which in turn implies that $x \in f(b_1 \vee b_2)$.

Therefore, $x \in f(b_1 \vee b_2) \Leftrightarrow x \in f(b_1) \cup f(b_2)$.

Preserving $\neg$: Let $x \in B$. Then $f(x) \cup f(\neg x) = f(x \vee \neg x) = f(\mathbf{1}) = X$ and $f(x) \cap f(\neg x) = f(x \wedge \neg x) =$

$f(\mathbf{0}) = \emptyset$. Thus $f(\neg x) = f(x)c$.

Corollary 8.3.16. *Let B be a finite Boolean algebra.*

1. If B has exactly k atoms then B is isomorphic to $P(\{1, 2, \ldots, k\})$. *Hence, B has exactly* 2^K *elements.*

2. Fix $b \in B$. *If* $p1, \ldots, pn$ *are the only atoms less than or equal to b, then b* $= p1 \vee \cdots \vee pn$

Check Your Progress 2

1. What is Boolean Isomorphism? Explain with example

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2. Define Induced partial order & atom.

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3.5 FINITE BOOLEAN ALGEBRAS AS N-TUPLES OF 0'S AND 1'S

The simplest nontrivial Boolean algebra is the Boolean algebra on the set B2 = {0, 1}. The ordering on B_2 is the natural one, $0 \leq 0$, $0 \leq 1$, $1 \leq 1$. If we treat 0 and 1 as the truth values "false" and "true," respectively, we see that the Boolean operations $\vee$ (join) and $\wedge$ (meet) are nothing more than the logical connectives $\vee$ (OR) and $\wedge$ (AND). The Boolean operation, – , (complementation) is the logical ¬ (negation). In fact, this is why the symbols –,$\vee$, and $\wedge$ were chosen as the names of the Boolean operations. The operation tables for $[B_2; -, \vee, \wedge]$ are simply those of "or," "and," and "not," which we repeat here:

$\vee$	0	1
0	0	1
1	1	1

$\wedge$	0	1
0	0	0
1	0	1

u	$\overline{u}$
0	1
1	0

3.6 BOOLEAN FUNCTIONS

A **Boolean Function** is described by an algebraic expression called **Boolean expression** which consists of binary variables, the constants 0 and 1, and the logic operation symbols. Consider the following example.

F (A, B, C, D)	=	$A + \overline{BC} + ADC$
Boolean Function		Boolean Expression

(1)

Here the left side of the equation represents the output Y. So we can state (1) as

$$Y = A + \overline{BC} + ADC$$

Truth Table Formation

A truth table represents a table having all combinations of inputs and their corresponding result.

It is possible to convert the switching equation into a truth table. For example, consider the following switching equation.

F(A, B , C) = A + BC

The output will be high (1) if A = 1 or BC = 1 or both are 1. The truth table for this equation is shown by Table (a). The number of rows in the truth table is 2^n where n is the number of input variables (n=3 for the given equation). Hence there are 2^3 = 8 possible input combination of inputs

Inputs			Output
A	B	C	F
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

Methods to simplify the boolean function

The methods used for simplifying the Boolean function are as follows −

- Karnaugh-map or K-map, and
- NAND gate method.

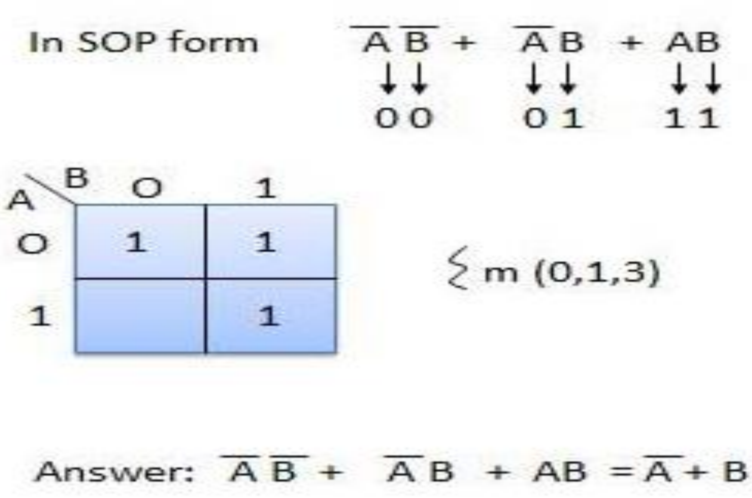
Karnaugh-map or K-map

The Boolean theorems and the De-Morgan's theorems are useful in manipulating the logic expression. We can realize the logical expression using gates. The number of logic gates required for the realization of a logical expression should be reduced to a minimum possible value by K-map method. This method can be done in two different ways, as discussed below.

Sum of Products (SOP) Form

It is in the form of sum of three terms AB, AC, BC with each individual term is a product of two variables. Say A.B or A.C etc. Therefore such expressions are known as expression in SOP form. The sum and products in SOP form are not the actual additions or multiplications. In fact they are the

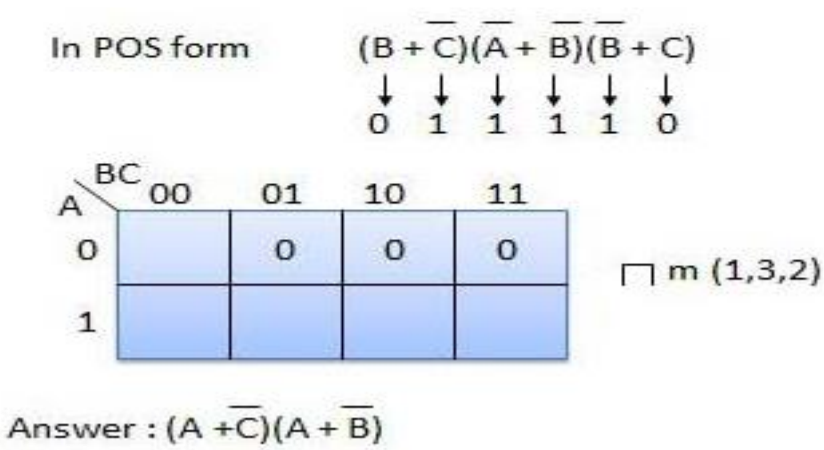
OR and AND functions. In SOP form, 0 represents a bar and 1 represents an unbar. SOP form is represented by



Product of Sums (POS) Form

It is in the form of product of three terms (A+B), (B+C), or (A+C) with each term is in the form of a sum of two variables. Such expressions are said to be in the product of sums (POS) form. In POS form, 0 represents an unbar and 1 represents a bar. POS form is represented by $\prod$.

Given below is an example of POS.

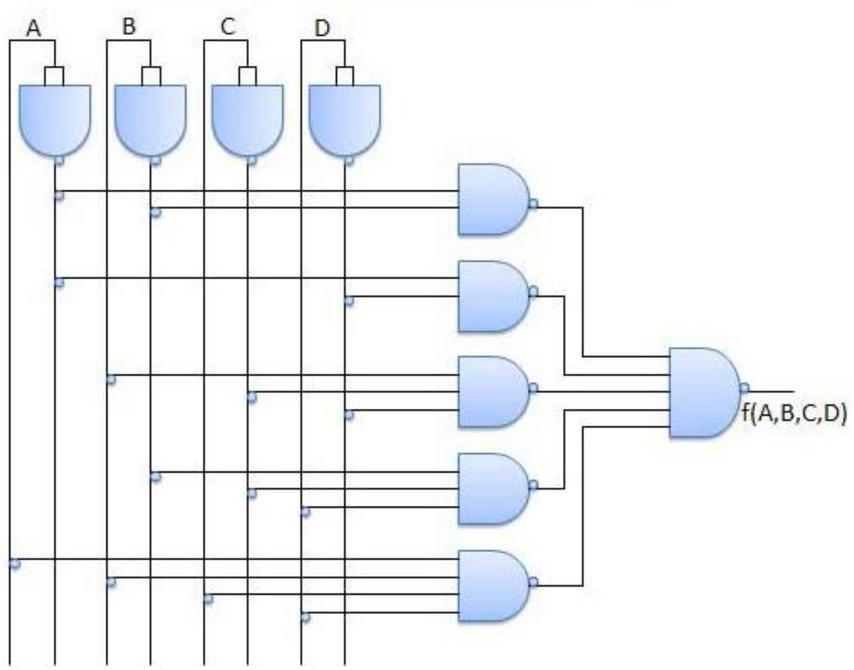


Notes

NAND gates Realization :

NAND gates can be used to simplify Boolean functions as shown in the example below.

$$F(A,B,C,D) = \overline{A}\overline{D} + ABCD + \overline{B}\overline{C}D + B\overline{C}\overline{D} + \overline{A}\overline{B}$$



Check Your Progress 3

1. Explain the methods to simplify the Boolean functions

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3.7 LET'S SUM UP

Boolean Algebra an essential tool since telephone, computers and many kinds of electronic control devices are based on a binary system, this branch of Mathematics are very useful for the internal working.

3.8 KEYWORDS

1. Boolean algebra, is a method for describing a set of objects or ideas

2. Truth table is a table, which represents all the possible values of logical variables/statements along with all the possible results of given combinations of values

3. Unary Operators: Unary operators are the simplest operations because they can be applied to a single True or False value. Identity: The identity is our trivial case. It states that True is True and False is False.

4. Negation: The negation operator is commonly represented by a tilde (~) or ¬ symbol. It negates, or switches, something's truth value.

3.9 QUESTIONS FOR REVIEW

1. What is the number of Boolean homomorphisms from P([4]) to P([3])?

2. Let B be a Boolean algebra. Then prove the following:
   (a) If B has three distinct atoms p, q and r, then p ∨ q 6= p ∨ q ∨ r.
   (b) Let b ∈ B. If p, q and r are the only atoms less than or equal to b, then b = p ∨ q ∨ r.

3. What are the atoms of the free Boolean algebra with generators p1, pn?

4. We know that a finite Boolean algebra must have at least one atom. Is 'finite' necessary?

5. Show that the set of subsets of N which are either finite or have a finite complement is a denumerable Boolean algebra. Find the atoms. Is it isomorphic to the free Boolean algebra with generators p1, p2, · · · ?

3.10 SUGGESTED READINGS

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3.11 ANSWER TO CHECK YOUR PROGRESS

1. [HINT: Provide definition – 3.2]
2. **[Provide the proof – 3.2.2]**
3. CONCEPT: A **Boolean isomorphism** is a Boolean homomorphism which is a bijection.—3.2 & provide example

4. DEFINITION: Let (B, ∨, ∧, ¬) be a Boolean algebra. The relation ≤ on B given by a ≤ b if and only if a ∧ b = a for all a, b ∈ Bis called the **induced partial order**.

A minimal element of B with respect to the partial order ≤, which is different from **0** is called an **atom** in B. –3.3 Provide examples & explain one proposition

5. State the types and explain anyone in details – 3.5.2

UNIT 4: RELATIONS AND DIAGRAPHS I

STRUCTURE

4.0 OBJECTIVES

- What is the concept of Digraphs
- Special Properties of Binary Relation
- Big O notation

- What is an Equivalence Relation?
- Ordering Relation, Partially Ordered sets, POSET diagram
- Enumeration, Enumeration Ordering &Lattices

4.1 CONCEPT OF DIAGRAPHS

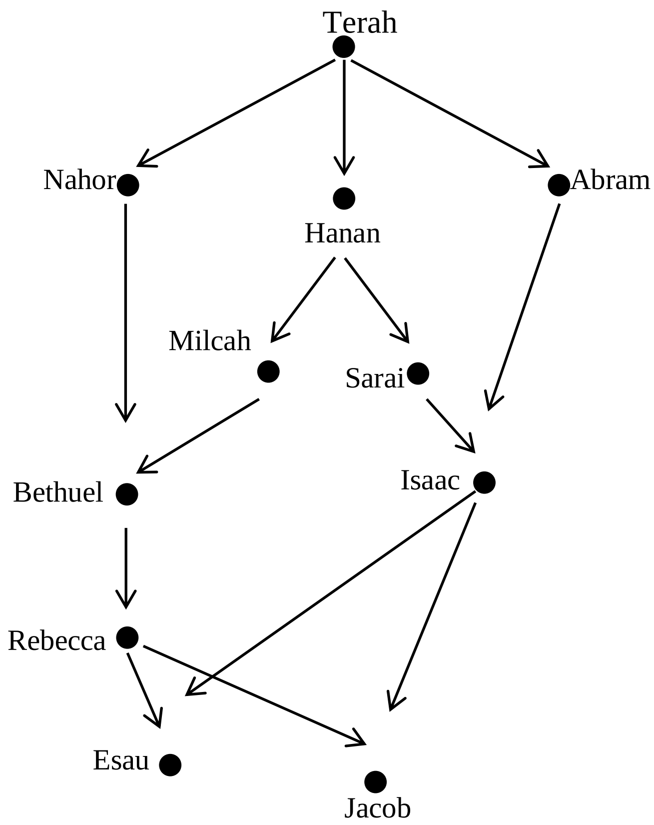


Figure 4.1

What is this diagram all about? At first sight it reflecting some name and arrows must be indicating some relation.

- The above diagram is known as directed graph / digraphs representing the kinship relation “is parent of” between eleven people. Each individual is represented by point, and an arrow is drawn from each parent to each of the respective children. Thus, Terah has three children.
- The binary relation represented by this directed graph is the set of pairs:

- { (Terah, Hanan), (Terah, Nahor), (Terah, Abram), (Hanan, Milcah), (Hanan, Sarai), (Abram, Issac), (Milcah, Bethuel), (Nahor, Bethuel), (Bethuel, Rebecca), (Sarai, Issac),(Issac, Jacob), (Issac ,Esau), (Rebecca, Esau), (Rebecca, Jacob)}
- So from both the diagram and the ordered pair, we can make out that diagram is easy to understand as compare to ordered pairs.

Concept:A pair of sets G = (V, E) is a **directed graph / digraph** if $E \subseteq V \times V$.The elements of V are called vertices and the elements of E are called edges. An edge (x, y) is said to be from x to y and is represented by an arrow with the tail at x and the head at y.

- Such an edge is said to be **incident from** x, **incident to** y, and **incident on** both x and y.
- If there is an edge in E from x to y we say x is **adjacen**t to y.
- The number of edges incident from a vertex is called **out-degree** of the vertex and the number of edges incident to a vertex is called the **in-degree.**
- A edge from a vertex to itself is called a **loop and will ordinarily permitted.**
- A digraph with no loops is called loop-free or simple.

[Note: Unless directed all digraphs are presumed to be finite; that is v is assumed to be a finite set.]

Example: With reference to Figure 6.1,

- ✓ The edge (Terah, Abram) is from Terah to Abram.
- ✓ There was **two edge incident** on Abram i.e. (Terah, Abram) **is incident to** Abram and (Abram, Issac) **is incident from** Abram.
- ✓ Terah has **out-degree three**.
- ✓ No other vertex has **in-degree** and **out-degree.**

[Note: For any digraph (V, E), E is a binary relation on V. Similarly any binary relation

$R \subseteq A \times B$ may also be viewed as a digraph G = (A ∪ B, R).]

- When more than one edge is permitted incident from one vertex to another vertex, then the result is a directed multigraph then the result is a directed multigraph and then two or more edges incident from a vertex x to vertex y are called **multiple edges.**
- A graph $G^1 = (V^1, E^1)$ is a sub graph of a graph $G = (V, E)$ if $V^1 \subseteq V$ and

  $E \subseteq E \cap (V * V)$. G is a proper subgraph of G if $G^1 \neq G$.

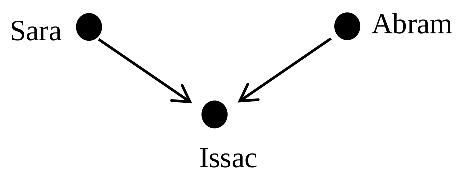


Figure 4.2

The above diagram comprises of vertices { Sarai, Abram, Issac} and the edges {(Sarai, Issac), (Abram, Issac)} is a proper sub graph of the digraph of figure 1.

- Two graphs $G_1 =(V_1, E_1)$ and $G_2 =(V_2, E_2)$ are isomorphic if there is a one-to-one onto function : $V_1 \rightarrow V_2$ that preserves adjacency. By preserving adjacency, we mean for digraphs that for every pair of vertices v and win V_1, (v, w) is in E_1 if and only if

  $[f (v), f (w)]$ is in E_2 or in other words,

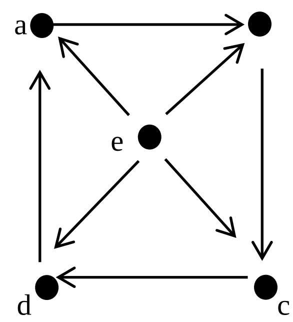
  $$E_2 =[f (v), f (w)] \mid (v, w) \in E_1$$

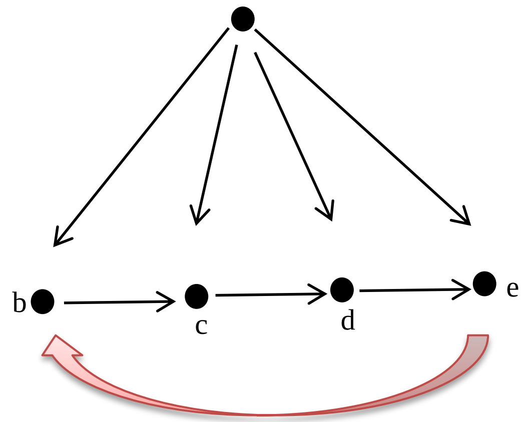
- In this case f a (directed graph) isomorphism from $G_1 \rightarrow G_2$.
- An invariant of graphs (under isomorphism) is a function g on graphics such that $g(G_1) = g(G_2)$ whenever G_1 and G_2 are isomorphic.

Example: The diagraphs in figure 3 below are isomorphic. They both have five vertices, eight edges, and degree spectrum (2, 1), (2, 1), (2, 1), (2, 1), (0, 4).

a

b





Example: Prove that each of the invariants cited in this section is truly an invariant:

a. The number of vertices
b. The number of edges
c. The degree spectrum.

Solution: Suppose that f is an isomorphism from $G_1 \rightarrow G_2$.

It follows from the definition of one-to-one onto function that / gives a one-to-one correspondence between the vertices of G_1 and G_2.

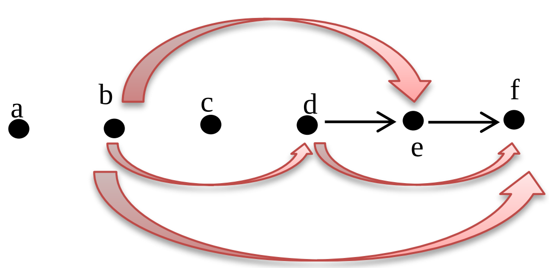
a. There is one-to-one correspondence between edges of G_1 and G_2 given by : $f[(x, y)] = [\ f(x)\ ,\ f(y)\ ]$
b. Suppose $v_1, \ldots, v_n$ is a list of the vertices of G_1 , ordered in decreasing order of in-degree, and within vertices of equal in-degree, by increasing out-degree.
   - For any v_i, $f(v_i)$ has the same in-degree in G_2 as v_i in G_1 and likewise for out-degree.
   - This is because every edges (v_i, v_j) in G_1 corresponds uniquely to an edge $[f(v_i)\ ,\ f(v_j)\ ]$ in G_2, and for every edge in G_2 there is such a corresponding edge in G_1.
   - Thus, the degree spectrum $\{$ in-degree (v_1), out –degree $(v_1)\}, \ldots,$ in-degree (v_n), out –degree $(v_n)\}$, must be identical to the degree spectrum

{ in-degree f (v_1), out –degree $f(v_1)$}, **…,** in-degree $f(v_n)$, out –degree $f(v_n)$}.

4.2 SPECIAL PROPERTIES OF BINARY RELATION:

We know following the following properties of a binary relation as:

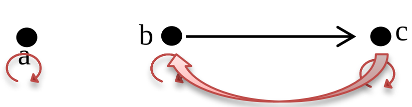
1. Transitivity $\forall x, y, z$ if $x\ R\ y$ and $y\ R\ z$, then $x\ R\ z$;
2. Reflexivity $\forall x$ $x\ R\ x$;
3. I reflexivity $\forall x$ $x \not R x$;
4. Symmetry $\forall x, y$ if $x\ R\ y$, then $y\ R\ x$;
5. Ant symmetry $\forall x, y$ If $x\ R\ y$ and $y\ R\ x$, then $x = y$;
6. Asymmetry $\forall x, y$ If $x\ R\ y$ then $y \not R x$



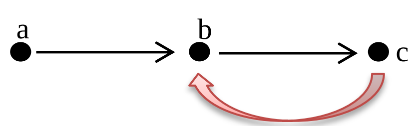
Transitive Relation

- A digraph is **transitive** if any of the three vertices like in above figure d, e, f, exhibits a relation like there is an edge from d to e, and an edge from e to f, also there is an edge from d to f.

[NOTE: d, e, and f in the above definition need not to be distinct.]

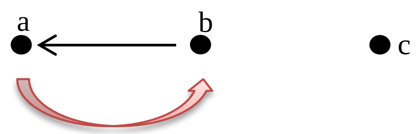


Reflexive Relation



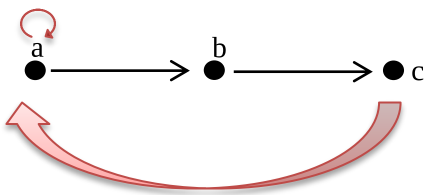
Irreflexive Relation

- A digraph is **reflexive** if every vertex has an edge from the vertex to itself (i.e. self-loop) and it is **irreflexive** if none of the vertices have self- loop. Refer the above figure



Symmetric Relation

- A digraph is **symmetric** if for every edge in one direction between points there is also an edge in the opposite direction between the same two points as illustrated in the above figure.



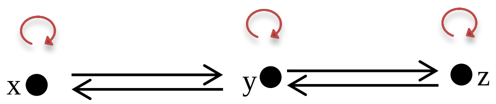
Antisymmetric Relation

- A digraph is **Antisymmetric** if no two distinct points have an edge going between them in both direction.
- A **asymmetric** digraph is further restricted, no self-loops are permitted.
- A binary relation that is transitive, reflexive and Antisymmetric is called a **partial ordering** relation.

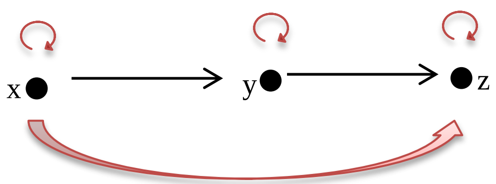
Example: Give an example of a nonempty set and a relation on the set that satisfies each of the following combinations of properties; draw a digraph of the relation.

a. Symmetric and transitive, but not reflexive

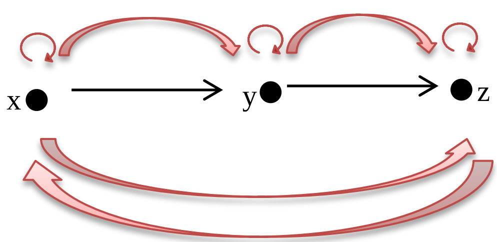
b. Symmetric and reflexive, but not transitive



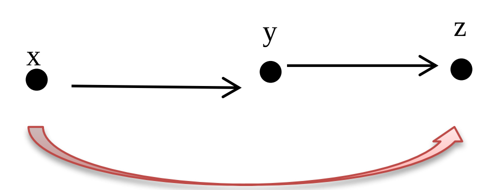
c. Transitive and reflexive, but not symmetric



d. Transitive and reflexive, but not Antisymmetric



e. Transitive and antisymmetric, but not reflexive



f.

g. Ant symmetric and reflexive, but not transitive.

4.3 BIG O NOTATION:

Concept: Let g: N → R be a function from the set of nonnegative integers into real numbers. O(g) denotes the collection of all functions f: N → R for which there exist constants c and k (possibly different for each f) such that for every $n \geq k$ $|f(n)| \leq c \cdot |g(n)|$. If f is in O(g) we say that f is **of order** g.

Lemma 1: If there exists a constant k, such that for every $n \geq k_1$, $f(n) \geq 0$ and $g(n) \geq 0$, then f is in O(g) if and only if there exist constants c and k_2, such that for every $n \geq k_2$, $f(n) \leq c \cdot g(n)$.

Proof: Since $f(n)$ and g(n) both are nonnegative for $n \geq k_1$, we have $|f(n)| = f(n)$ and $|g(n)| = g(n)$ for $n \geq \max(k_1, k_2)$. The lemma then follow immediately from the above concept.

Example 1: Consider the functions $f(n) = 2^n$ and $f(n) = 3^n$. Since $3^n \geq 2^n$ for all $n \geq 0$, we know that f is in O(g) and g is not in O(g).

Suppose g is in O(g) then there exist c and k such that for all positive $n \geq k$, $3^n \leq c \cdot 2^n$ which implies a contradiction

$$n \leq \frac{\log_e c}{\log_e (3/2)}$$

Example 2: Consider the functions

$$f(n) = \begin{cases} 2^n & \text{if } n \text{ is an even integer} \\ n & \text{otherwise} \end{cases}$$

and

$$g(n) = \begin{cases} 2^n & \text{if } n \text{ is an odd integer} \\ n & \text{otherwise} \end{cases}$$

these functions are pathological, as their definitions might lead one to suspect. Suppose that f is in O(g). Then for some c and k it would be true that for all (positive) $n \geq k$, $f(n) \leq c \cdot g(n)$. Taking n to be even, this would mean that $2^n \leq c \cdot n$, which is a contradiction. Similarly, for the case when n is odd, show that g cannot be in O (f).

[NOTE: We can avoid considering special names, like f and g for the functions being compared.]

2^n is in O (3^n) , but 3^n is not in O(2^n).

For example , since $1^2 + 2^2 + \ldots + n^2 = (1 / 3)\, n\, (n + 1 / 2)\, (n + 1) = 1/3\, n^3 +$ O (n^2) stands for $g(n) = 1 / 2\, n^2 + 1 / 6\, n$, which is specific function g in O(n^2).

THEOREM: The relation Q = {(f, g) | f : N→ R, g : N →R, Cis in O(g) } is reflexive and transitive, but is not a partial ordering or an equivalence relation.

PROOF: We must prove four things

a. Q is reflexive
b. Q is transitive
c. Q is not Antisymmetric
d. Q is not symmetric

a. Since $| f(n) | \leq 1. | f(n) |$ for all $n \geq 1$, we know that (f , f) is in q for all f: R → R, and so Q is reflexive.
b. Suppose (f, g) and (g, h) are in Q. Then there exist c_1, c_2, k_1 and k_2 such that for all $n \geq$,
$| f(n) | \leq c_1 . |g(n)|$,and forall $n \geq k_2$, $| g(n) | \leq c_2 . | h(n) |$. It follows that for all n greater than or equal to the maximum k_1 and k_2, $| f(n) | \leq c_1 . | g(n) | \leq c_1 . c_2 . | h (n) |$, and so Q is transitive.
c. It is also possible to show that to have(f, g) and (g, f) being in Q.
d. Previous example 1 shows that it is possible to have (f, g) in Q without (g, f) being in Q.

Example: Prove or disapprove each of the following:

a. If f is in O(g) and c is a positive constant, then c . f is in O(g)

Suppose for all $n > k$, $| f(n) | < a \cdot | g(n) |$. Then $c \cdot | f(n)| \leq c \cdot a | g(n) |$

b. If f_1 and f_2 are in O(g) then $f_1 \cdot f_2$ then $f_1 + f_2$ is in O(g)

The crucial fact here is that $| x + y| \leq | x | + | y |$.

Suppose for all $n > k_1$, $| f_1(n) | < a_1 \cdot | g(n) |$ and

for all $n > k_2$, $| f_2(n) | < a_2 \cdot | g(n) |$.

Then for all $n > \max(k_1, k_2)$,

$| f_1(n) + f_2(n) | \leq | f_1(n) | + | f_2(n) | \leq a_1 \cdot | g(n) | + a_2 \cdot | g(n) | \leq (a_1 + a_2) \cdot | g(n) |$.

c. If f_1 is in $O(g_1)$ and f_2 is in $O(g_2)$ then $f_1 \cdot f_2$ is in $O(g_1 \cdot g_2)$

The crucial fact here is that $| x \cdot y| = | x | \cdot | y |$.

Suppose for all $n > k_1$, $| f_1(n) | \leq a_1 \cdot | g_1(n) |$ and

for all $n > k_2$, $| f_2(n) | \leq a_1 \cdot | g_2(n) |$.

Then for all $n \geq \max, (k_1, k_2)$,

$| f_1(n) \cdot f_2(n) | \leq | f_1(n) | \cdot | f_2(n) | \leq a_1 \cdot | g_1(n) | \cdot a_2 \cdot | g_2(n) | \leq (a_1 \cdot a_2) \cdot | g_1(n) \cdot g_2(n) |$

d. If f_1 is in $O(g_1)$ and f_2 is in $O(g_2)$ then $f_1 + f_2$ is in $O(g_1 + g_2)$

This is true for positive valued functions, but it is false if we consider negative valued function.

Consider $f(x) = x$, $g_1(x) = x^2$ and $g_2(x) = - x^2$.

Clearly, f is in $O(g_1)$ and in $O(g_2)$,due to the absolute values, but $O(g_1 + g_2) = O(0)$, and f cannot be in O(0).

CHECK YOUR PROGRESS 1

1. What is Digraph? Explain the transitive property of Digraphs.

2. Explain the Big O notation

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4.4 EQUIVALENCE RELATION:

- Consider a general example, suppose we want to buy a gift from a departmental store which cost Rs. 50. It wouldn't matter to the cashier of the departmental store to accept Rs. 50 , credit or debit card or a credit note of Rs. 50 or gift coupon of Rs. 50, as he will be accept any of this. That means these all types are equivalent in purchasing power.
- We will consider other situation, suppose for example Rs. 50 note is lighter than the sum of coins that is equal to Rs. 50. Suppose we have tostore it in a bag then obviously weight of bag containing coins whose sum is Rs. 50 would be heavier than a bag containing Rs. 50 note. These two bags are not equivalent in terms of weight.
- The meaning of equivalent depends on the context and expresses the notion of being the same in those respects relevant to the context.
- A binary relation is an equivalence relation if it is transitive. For example, consider the relation " was born in the same month as" This relation is clearly reflexive, since each individual was born in the same month as himself.
- It is equally clearly symmetric – if individual A was born in the same month as individual B, then B was born in the same month as A.
- There is no question about transitivity. A and B were born in the same month and if B and C were born in the same month, no one is likely to deny that A and C must also have born in the same month.
- Equivalence relation can also be considered in another way of dividing things into classes.
- Consider the same example as above "was born in the same month as "

- Partition the set of al living human beings into twelve disjoint classes, corresponding to the twelve months of the year. Each of these equivalence classes consists of all the people who were born in a given month like November.
- Any time a set is partitioned into disjoint non empty subsets an equivalence relation is involved
    - The two notions i.e. treating a case in terms of relation and in termsof partition are interchangeable.

CONCEPT: Given a set A, a partition of A is a collection P of disjoint subsets whose union is A. That is

1. For any $B \in P$, $B \subseteq A$;
2. For any $B, C \in P$, $B \cap C = \emptyset$, or $B = C$; and
3. For any $x \in A$ there exist $B \in P$, such that $x \in B$.

Concept: Given any set **A** and any Equivalence relation **R** on **A**, the equivalence class [x] of each element **x** of **A** is defined $[x] = \{ y \in A \mid \mathbf{x\ R\ y} \}$

[NOTE: we can have [x] = [y], even if x = y, provided **x R y**]

THEOREM: Given any set **A** and any equivalence relation **R** on **A**, $S = \{[x] \mid x \in A\}$ is a partition of **A** into disjoint subsets. Conversely , if **P** is a partition of **A** into nonempty disjoint subsets, then **P** is the set of equivalence classes for the equivalence relation **E** defined on **A** by **a E b** if and only if a and b belongs to the same subset of **P**.

PROOF:

1. Clearly by the definition of [x], $[x] \subset A$.
2. $[x] \cap [y] = \{ z \in A \mid \mathbf{x\ R\ z} \text{ and } \mathbf{y\ R\ z}\}$. If this set is not empty, then for some $z \in A$, **xR z** and **y R z**; but then, since R is transitive and symmetric, **x R y**, so that [x] = [y].
3. For any $x \in A$, $[x] \in S$.

Example 1: Tell how many distinct equivalence classes there are for each of the following equivalence relations.

a. Two people are equivalent if they are born in the same week.
   53 – as one class for each week

b. Two people are equivalent if they are born in the same year.
   One class for each year in which a person was born, possibly an unbounded number.

c. Two people are equivalent if they are of the same sex.
   Two, one for males and other for females.

Example 2: Suppose **R** is an arbitrary transitive reflexive relation on a set A. Prove that the relation **E** defined by x E y is an equivalence relation on **A** if and only if x R y and y R x.

Solution:

Transitivity: if x E y and y E z, then x R y and y R x , y R z and z R y.

Since R is transitive, this means x R z and z R x which means x E z.

Reflexivity: Since R is reflexive, x R x, and so also x E x.

Symmetry: If x E y, then x R y and y R x, which means y E x.

Example 3: Let S={1,2,3,4,5,6} and A1 ={3,6} A2 ={1,4} A3 = {2,5}

a. Is A1, A2,, A3 a partition of S ? Yes.
b. Give a partition of S?
   {,2,4,6} {1,3,5}
   {1,2} {3,4,5} {6}

4.4.1 Compatibility Relations:

If a relation is only reflexive and symmetric then it is called a compatibility relation. A table shown below represents a compatibility relation. So, every equivalence relation is a compatibility relation, but not every compatibility relation is an equivalence relation.

	x	**y**	**Z**
x	1	1	1
y	1	1	--
z	1	--	1

4.5 THE INTEGERS MODULO M:

- One of the important applications of Equivalence Relation in Computer Science, is modular arithmetic.
- It is required because of finite storage limitations and finite accuracy limitations of hardware arithmetic operations on computers.
- We can take the simple example of our 12- hour clock, which counts seconds and minutes modulo 60 and hours 12.

CONCEPT: Let $\boldsymbol{m}$ be any positive integer. The relation congruence modulo m [written $\equiv$ (mod $\boldsymbol{m}$)], is defined on the integers by $x \equiv$ (mod $\boldsymbol{m}$) if and only if $x = y + a \,\boldsymbol{.}\, \boldsymbol{m}$ for some integer a.

THEOREM: For any positive integer $\boldsymbol{m}$**,** the relation$\equiv$ (mod $\boldsymbol{m}$) is an equivalence relation on the integers, and partitions the integers into $\boldsymbol{m}$ distinct equivalent classes: [0], [1], **… ,** [$\boldsymbol{m}$ – 1].

PROOF: If **x** is any integer, the division algorithm implies $x = mq + r$, where **q** and **r** are integers and $0 \le r < m$. thus, $x \equiv r \bmod m$ and $[x] = [r]$. Thus, each equivalence class for this relation is one of the classes [0] , [1], **…**, [m – 1].

Morever, if [x] = [y] where $0 \le x \le y \le m - 1$, then $y = ma + x$ for some integer **a**.

Therefore, $0 \le y - x = ma < m$ implies $a = 0$ and $x = y$.

The equivalence class [r] is frequently called a congruence class, and the collection of congruence classes [0], [1], **…**, [m -1] of integers with respect to the relation ≡ (mod m) is customarily denoted by Z_m, for any positive integer m. That is, Z_m = { [0], [1], **…,** [m -1] }. Arithematic on the integers can be extended to arithmetic on Z_m in a natural way:

$$[x] + [y] = [x + y];$$
$$-[x] = [-x];$$
$$[x] . [y] = [x . y]$$

THEOREM: The operations +, – , and **.** on Zmare well defined functions.

PROOF:

1. Suppose $x_1 \equiv x_2$ (mod m) and $y_1 \equiv y_2$ (mod m). We need to show that

$x_1 + y_1 = x_2 + y_2$ (mod m).

By the definition of ≡ (mod m), we know that $x_1 = x_2 + a . m$ for some integer a, $y_1 = y_2 + b . m$ for some b.

It follows that $x_1 + y_1 \equiv (x_2 + y_2) + (a + b) . m$, so that $x_1 + y_1 = x_2 + y_2$ (mod m).

2. Suppose that $x_1 \equiv x_2$(mod m). Then $x_1 = x_2 + a . m$ for some a, and $-x_1 = -x_2 + (-a) . m$, so that $[-x_1] = [-x_2]$.

3. Suppose that $x_1 \equiv x_2$ (mod m) and $y_1 \equiv y_2$ (mod m).
   Then $x_1 = x_2 + a . m$ for some integer a, $y_1 = y_2 + b . m$ for some b.
   It follows that $x_1 . y_1 = x_2 . y_2 + [x_2 . b + y_2 . a + a . b . m] . m$, so that $x_1 . y_1 \equiv x_2 . y_2$ (mod m).

As the operation addition, subtraction and multiplication holds true so we will have following laws:

1. [x] + [y] = [y] + [x] Addition is commutative
2. [x] **.** [y] = [y] **.** [x] Multiplication is commutative
3. ([x] **+** [y]) + [z] = [x] + ([y] + [z]) Addition is Associative
4. ([x] **.** [y]) **.** [z] = [x] **.** ([y] **.** [z]) Multiplication is Associative
5. ([x] **+** [y]) **.** [z] = [x] **.** [z] + [y] **.** [z] Multiplication distributes over addition

CONCEPT:

- Let **x** and **y** be integers. Recall that **xdividesy** if there exist an integer **z** such that x . z = y.

- The **greatest common divisor** of the two positive integers is the largest positive integer that divides both of them. The notation gcd (x , y) denotes the greatest common divisor of **x** and **y**.

- Two integers are **relatively prime** if their greatest common divisor is 1.

Example: The greatest common divisor of 237 and 204 is 3. Also, gcd(237, 158) = 79, and gcd (237, 203) = 1. Thus, 237 and 203 are relatively prime.

THEOREM: If **x** and **m** are relatively prime positive integers then, for every positive integer w, the equivalence classes [w], [w + x], [w + 2 **.** x], **…**, [w + (m – 1) **.** x] are all distinct.

First we will prove the Lemma (lemma is a small results) :

Suppose **x** and **m** are positive integers and **r** is the smallest positive integer for which there exist integers **c** and **d** such that r = c **.** x **+** d **.** m. Then r = gcd (x, m).

PROOF:

- We will first show that r divides x. Suppose $x = p \cdot r + q$, where $0 \leq q < r$. (That is, q is the remainder when x is divided by r.) then,
    - $q = x - p \cdot r = x - p \cdot (c \cdot x + d \cdot m)$
        - $= (1 - p \cdot c) \cdot x + (-p \cdot d) \cdot m$

- Since r is the smallest positive integer of this form, and $0 \leq q < r$, it must be that $q = 0$, which is to say r divides x.
- If we interchange x and m in the above argument, it will gives us results as r also divides m.
- Now to prove that r is the largest positive integer that divides both x and m, we will consider another integer which divides both x and m, which is less than or equal to r.
- Suppose s is a positive integer that also divides x and m. then, for some a and b, $x = a \cdot s$ and $m = b \cdot s$.
- Substituting $a \cdot s$ for x and $b \cdot s$ form in $r = c \cdot x + d \cdot m$, we will get $r = (c \cdot a + d \cdot b) \cdot s$.
- As r and s are both positive, $c \cdot a + d \cdot b$ is also positive
- This means that $r \geq s$, since if $r < s$ then $c \cdot a + d \cdot b < 1$, which would be contradiction.
- Observe that the set S of all positive integers y such that $y = xs + mt$, for integers s and t, is a nonempty set since $x^2 + m^2$ is in S. Therefore, by the well ordering property of the positive integers, there exists a minimal element r in S. By above Lemma, r = gcd (x, m)

PROOF OF ABOVE STATED THEOREM:

Suppose $[w + x \cdot i] = [w + x \cdot j]$ and $0 \leq j < i < m$.

For some integer y, $w + x \cdot i = w + x \cdot j + y \cdot m$. Cancel out the w-term and combine the x-term, we get,

1. $x \cdot (i - j) = y \cdot m$ Since x and m are relatively prime, gcd (x, m) = 1, and , by the preceding lemma, there exist c and d such that
2. $c \cdot x + d \cdot m = 1$ From (1) we obtain

3. $c \cdot x \cdot (i - j) - c \cdot y \cdot m = 0$ From (2) we obtain
4. $c \cdot x \cdot (i - j) + d \cdot m \cdot (i - j) = (i - j)$ Subtract (3) from (4) we obtain,
5. $m \cdot [d \cdot (i - j) + c \cdot y] = i - j$

This is a contradiction, since $0 < i - j < m$ implies $0 < d\,(i - j) + c \cdot y < 1$, and there is no integer between zero and one .

Example 1: Let set A={ 2, 4, 6, 8, 10 } and R be a binary relation on A defined by

$$(m,n) \in R \quad \text{if and only if} \quad m \equiv n \pmod 4 \quad \forall\, m,n \in A$$

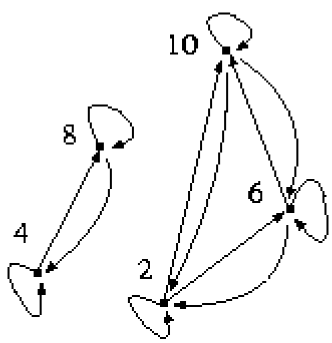
(m,n)	m-n	Conclusion	Is multiple of 4?
(2,2)	*2-2=0*	*(2,2)* ∈ *R*	yes, 0 is a multiple of 4
(2,4)	*2-4=-2*	*(2,4)* ∉ *R*	no, -*2* is not a multiple of 4
(2,6)	*2-6=-4*	*(2,6)* ∈ *R*	-*4*: yes
(2,8)	*2-8=-6*	*(2,8)* ∉ *R*	-*6*: no
(2,10)	*2-10=-8*	*(2,10)* ∈ *R*	-*8*: yes

(4,2)	*4-2=2*	$(4,2) \notin R$	*2*: no
(4,4)	*4-4=0*	$(4,4) \in R$	*0*: yes

which gives explicitly

$$R = \{ (2,2), (2,6), (2,10), (4,4), (4,8), (6,6), (6,10), (8,8), (10,10), (6,2), (10,2), (8,4), (10,6)\} .$$

This relation *R* can be drawn as



and is obviously an equivalence relation. Note that there are 2 ``connected" components, one containing elements 4 and 8 and the other, elements 2, 6 and 10. Here the ``connection" is made through certain walks *along* the directions of the arrows. These 2 components are just the 2 distinct ***equivalence classes*** under the equivalence relation *R*. Naturally we may use any element in an equivalence class to represent that particular class which basically contains all elements that are connected to the arbitrarily chosen representative element. Hence, in this case, we may choose 2 to represent the class *{2,6,10}*, written simply *[2]={2,6,10}*, and choose for instance 8 to represent the other class *{4,8}*, that is, *[8]={4,8}*. Since any member of an equivalence class can be used to represent that class, *[6]* and*[10]* will be representing exactly the same equivalence class as *[2]*. Hence

$$[2] \equiv [6] \equiv [10] = \{2, 6, 10\}, [4] \equiv [8] = \{4, 8\}$$

are the 2 distinct equivalence classes.

Example 2: Let set A={ 2, 4, 6, 8, 10 } and R be a binary relation on A defined by

$$(m,n) \in R \quad \text{if and only if} \quad m \equiv n \pmod 4 \quad \forall\, m,n \in A$$

Solution:

a. As per the definition of equivalence classes, it implies $2 \in [2]=\{2,6,10\}$, $6 \in [6]$, $10 \in [10]$, $4 \in [4]$ and $8 \in [8]=\{4,8\}$

b. In this case is the same as saying [2]=[6]=[10] and [4]=[8],

c. This case is consistent with the fact that any 2 of the equivalence classes [2], [4], [6], [8] and [10] are either disjoint (e.g. [2] and [4] have no elements in common) or exactly the same (e.g. [2]=[6]).

d. For the equivalence class {2,6,10} implies we can use either 2 or 6 or 10 to represent that same class, which is consistent with [2]=[6]=[10] . Similar observations can be made to the equivalence class {4,8}.

Example 3: Show that the distinct equivalence classes in example1 form a partition of the set A there.

Solution In example 4 we have shown that [2] ={2,6,10} and [4]={4,8} are the only distinct equivalence classes. Since A in example 4 is given by A ={2,4,6,8,10}, we can easily verify

(a): $A = [2] \cup [4]$; (b): $[2] \cap [4] = \emptyset$.

From the definition of the set partition we conclude that {[2], [4]} is a partition of set A.

Check Your Progress 2

1. Define the following terms

   a. Greatest common divisor

   b. Relative prime

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2. What is compatibility relations

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3. State the condition important for equivalence relations

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4.6 ORDERING RELATIONS

4.6.1 Poset Diagram

- A relation R on a set A is called a **partial order** on A when R is reflexive, antisymmetric, and transitive, and then the set A is called a **partially ordered set or a poset**.

- [A ; R] is used to denote A is partially ordered by the relation R .
- '≤'-It represent an arbitrary partial order on A .

- The characteristic properties of a partial order can be described as follows:

- $\forall$ a ϵ A, a $\leq$ a

  (Reflexivity)
- $\forall$ a, b ϵ A, if a $\leq$ b and b $\leq$ a, then a = b

  (Antisymmetry)
- $\forall$ a, b, c ϵ A, a $\leq$ b and b $\leq$ c, then a $\leq$ c

  (Transitivity)

- Two elements a and b in A are said to be **comparable** under $\leq$ if either a $\leq$ b or b $\leq$ a; otherwise they are **incomparable**.

- If every pair of elements of A are comparable, then we say [A; $\leq$] is **totally ordered** or that A is **totally ordered set or a chain**. In this case, the relation $\leq$ is called a **total order.**

Example:

1. Let U be an arbitrary set and let A = P (U) be the collection of all subsets of U.$\subseteq$

   Then [P (U);__] is a poset but if U contains more than one element, then P(U) is not totally ordered under set inclusion.

   If U contains the two distinct elements a and b, then P(U) contains two distinct elements {a} and {b} and these sets are incomparable under inclusion.

2. If Z is the set of integers and $\leq$ is the usual ordering on Z, then not only is [Z; $\leq$] partially ordered, but , more than that, it is totally ordered.

3. Another familiar poset involves the set P of positive integers and the relation "divides" where we write x | y or if and only if x divides y or if and only if y = xz for some integer z. Then [P;|] is a partially ordered set.

POSET DIAGRAMS:

- On a poset diagram there is a vertex for each element of A, but beside that, all loops are omitted eliminating explicit representation of the reflexive property.

- An edge is not present in poset diagram if it is implied by the transitivity of the relation.

- If we write x < y to mean x ≤ y but x = y, then an edge connects a vertex x to a vertex y if and only if y covers x, that is if and only if there is no element z such that x < z and z < y.

4.6.2 Hasse Diagram:

Hasse diagrams are meant to present partial order relations in equivalent but somewhat simpler forms by removing certain deducible ``noncritical" parts of the relations.

To construct a Hasse diagram:

1) Construct a digraph representation of the poset *(A, R)* so that all arcs point up (except the loops).

2) Eliminate all loops

3) Eliminate all arcs that are redundant because of transitivity

4) Eliminate the arrows at the ends of arcs since everything points up.

Example: Construct the Hasse diagram of *(P({a, b, c}),⊆).*

Solution: The elements of *P({a, b, c})* are

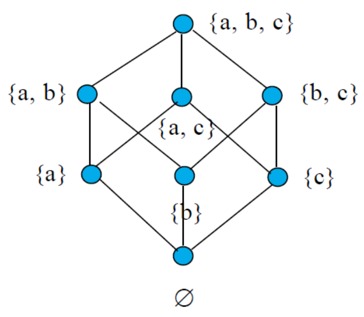
∅

{a}, {b}, {c}

{a, b}, {a, c}, {b, c}

{a, b, c}

The digraph is



4.6.3 Special Elements In Poset:

Let [A; ≤] be a poset and let B be a subset of A. Then

1. An element b ∈ B is called the **least element** of B if b ≤ x for all x ∈ B. The set B can have at most one least element. For if b and b' were two least elements of B, then we would have b ≤ b' and b'≤ b. Hence, by Antisymmetry b = b'.

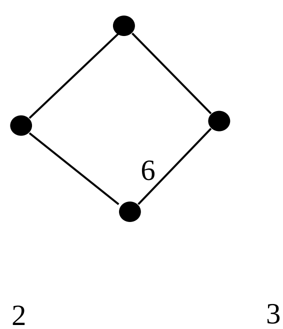
   An element b∈ B is called the **greatest element** of **B** if x ≤ b for all x ∈ B. The set **B** can have at most one greatest element.

2. An element b ∈ B is a minimal (maximal) element of **B** if x < b (x > b) for no x in **B**.If the set **B** contains a least element **b**, then of course **b** is the only minimal element of **B**. However, if the set **B** contains a minimal element, it need not be the only minimal element of **B**.

3. An element b ∈ A is called a **lower (upper) bound** of **B** if b ≤ x (b ≥ x) for all x ∈ B.

4. If the set of lower bounds of **B** has a greatest element, then this element is called the **greatest lower bound (glb)** of B;
   Similarly if the set of upper bounds of B has a least element , then this element is called the **least upper bound (lub)** of B.

Example: We will consider the following figures of Posets.



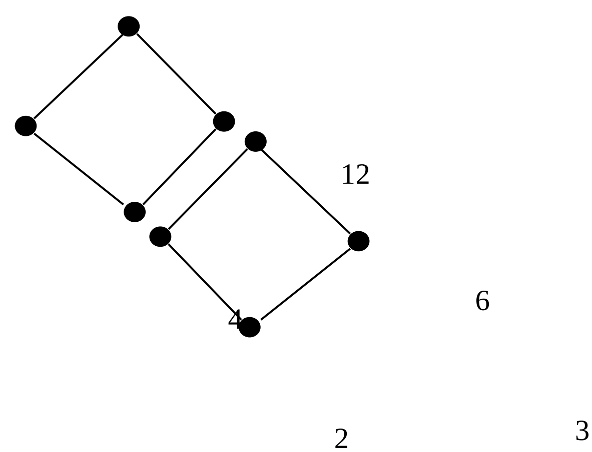


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Fig. a

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Fig. b

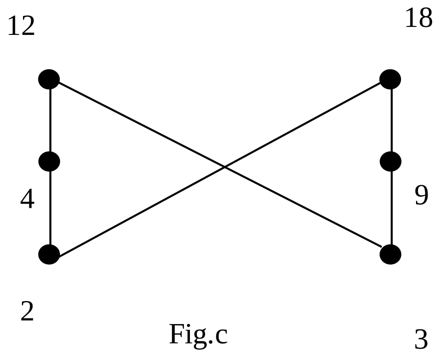

Fig.c

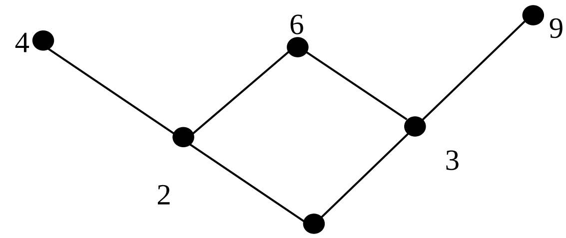

Fig.d

- Fig. a, b and d have a unique least element.
- The poset of c have several minimal elements namely, 2 and 3
- The poset of a and b have a unique greatest element
- The poset of c has two maximal elements 12 and 18
- The poset of d has maximum elements as 4, 6 and 9.

Maximal and Minimal Elements

Definition: Let *(A, R)* be a poset. Then *a* in *A* is a *minimal element* if there does not exist an element *b* in *A*such that *bRa*.

Similarly for a *maximal element.*

Note: There can be more than one minimal and maximal element in a poset.

Example: In the above Hasse diagram, ∅ is a minimal element and *{a, b, c}* is a maximal element.

4.6.4 Well Ordered Sets:

- A total order ≤ on a set A is well order if every nonempty subset B of A contains a least element. Moreover, [A; ≤] is said to be well ordered.
- Since A is totally ordered set it follows that if a set B contains a minimal element, then B contains only one element and moreover, this element is a least element of B.

Examples:

1. The poset [N; ≤], where **N** is the set of nonnegative integers, is well ordered. In actual fact, the well ordering property of ≤ is equivalent to the principle of mathematical induction, and is usually taken as an axiom.

2. On the other hand, the poset [Z; ≤] is not well ordered because the subset of negative integers does not contain a minimal element.

3. The relation ≤ on the set **Q** of rational numbers is a total ordering but not a well ordering because some subsets of **Q** do not contain a minimal element. For example, the set **P** of positive rational numbers contains no minimal element for if $x \in P$, $x / 2 < x$, and $x /2 \in P$.

4. Any finite totally ordered set is well ordered.

Check Your Progress 3:

1. What is Hasse Diagram? Enumerate with example

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2. Explain the terms
   a. Minimal Element
   b. POSET diagram

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4.7ENUMERATIONS:

Example: A well order on the set **Z** of integers can be constructed by listing the elements of N in ascending order and then pairing the elements of **Z** with those in **N** in a one-to-one correspondence like

f: N → Z defined as follows:

N:	0	1	2	3	4	5	6
	↓	↓	↓	↓	↓	↓	↓
Z:	0	-1	1	-2	2	-3	3

Pair the even integers in **N** with the positive integers of **Z** and the odd integers of **N** with the negative integers of **Z**. Then we define a new ordering **R** of **Z** by **a R b** if and only if $f^{-1}(a) \leq f^{-1}(b)$. Thus, **(– 1) R (– 3)** because 1 < 5.

The ***f*** above is a special case of an enumeration which we will define below as :

CONCEPT:

- Let **I** be an 'initial segment' of the non-negative integers. That is let I = {k | k ϵ N, k ≤ n} for some constant **n,** or let I = N.

- A function f: I → S is an enumeration of **S** if ***f*** is onto; that is, for repetition if the function is one-to-one, that is f (i) = f (j) only if i = j.

- Any set that has an enumeration is said to be countable. Sets that do not have enumeration are said to be uncountable or nondenumerable.

- The real numbers form a set which is totally ordered by ≤ but uncountable.

- Concept of enumeration without repetition and well ordering are closely related.

- Suppose f: I → A is an enumeration of A. For each **a** in **A**, let g(a) be the smallest integer in I such that f (n) = a.

- The relation $\leq_f$ defined by a $\leq_f$ b if and only if g(a) ≤ g(b) is a well ordering of A for which each element a ∈ A has only finitely many elements b ∈ A such that b$\leq_{f.}$

- Conversely, if **R** is a well ordering of a countable set A for which each a ∈ A has only finitely many predecessors; that is, there are only finitely many elements b ∈ A such that (b, a) ∈ R, then there is a unique enumeration without repetition f such that $\leq_f$ = R. This enumeration is given by $f(0)$ = min (A), $f(i)$ = min (A –{ $f(j)$ | j < i}).

4.8 LATTICES:

Join-semi lattice:- as a posset [A; ≤] in which each pair of elements a and b of A have a least upper bound. We also call this as **lub** the join of **a** and **b** and is represented as a ∨ b.

Meet-semi lattice :- as a posset in which each pair of elements **a** and **b** have a greatest lower bound; this **glb** is called the meet of a and b, and it is denoted by a $\wedge$b.

Thus, if c = a $\wedge$b. , then c satisfies:

1. c $\leq$ a and c $\leq$ b (c is a lower bound of {a, b})
2. If d $\leq$ a and d $\leq$ b, then d $\leq$ c (c is the greatest lower bound of {a, b})

Similarly, if c = a$\vee$b. then c should satisfies two similar properties as stated above after reversing the inequalities and changing the words lower bound to upper bound.

CONCEPT: A **lattice** is a poset in which each pair of elements has a least upper bound and a greatest lower bound. In other words, a lattice is both a join semilattice and a meet-semilattice.

Example:

1. If **U** is any set, [P (U) ; $\subseteq$] is a lattice in which the least upper bound of two subsets **B** and **C** of U is just B $\cup$ C and the greatest lower bound of {B, C} is B $\cap$C.

   PROOF: We know that B $\cap$ C $\subseteq$ B and B $\cap$ C $\subseteq$C so that B $\cap$ C is a lower bound of {B, C}.

   On the other hand, if D $\subseteq$ B and D $\subseteq$C, then D $\subseteq$ B $\cap$ C. Thus, B $\cap$ C is the greatest lower bound of {B,C}.

   Similarly we get B $\cup$ C is the lub of {B, C}

2. Any totally ordered set is a lattice in which a $\vee$ b is simply the greater and a $\wedge$ b is the lesser of a and b. For example, If R is the set

of real numbers with the usual ordering $\leq$, then a $\vee$ b = max {a , b} and a $\wedge$ b = min {a, b}.

3. The poset [P; |], where P is the set of positive integers, is a lattice in which a $\wedge$ b = gcd (a, b) and a$\vee$ b = lcm (a, b) where gcd and lcm respectively stand for greatest common divisor and least common multiple. For instance, 6 $\wedge$9 = 3 and 6 $\vee$9 = 18.

Example:

1. Find the glb and lub of the sets {3,9,12} and {1,2,4,5,10} if they exist in the poset (Z+,|).
   - glb=3 glb = 1
   - lub=36 lub = 20

4.9APPLICATION: STRINGS AND ORDERING ON STRINGS:

Let $\sum$ be any finite set. A finite sequence of zero or more elements chosen from $\sum$ is a string over $\sum$. In this context, $\sum$ is called an alphabet. The length of string w is denoted by |w|. The string of the length 0 is denoted by $\Lambda$and called the null string.

The set of all strings of length k is denoted by $\sum^k$. That is,

$\sum_0 = \{ \Lambda\}$, and

$\sum^{k+1} = \{w a| w \in \sum^k$ and $a \in \sum\}$ for $k \geq 0$.

$\sum^* \cup_{k\geq 0} \Sigma^k$, denoting the set of all strings over $\sum$.

$\Sigma^+ \cup_{k>0} \Sigma^k$, denoting the set of all non-null strings over $\sum$.

Thus, for every $w \in \sum^k$, $|w| = k$.

[wy denotes the **catenation(**Arrange in a series of rings or chains**)** of strings w and y].

If $w = w_1 \ldots w_n$ and $y = y_1 \ldots y_m$, $wy = w_1 \ldots w_n y_1 \ldots y_m$)

4.9.1 Lexicographic Order

Example : If $\sum = \{A, E, I, O, U\}$, $\sum^*$ includes all the words that can be written with vowels and in particular $\sum^6$ includes the string "VOWELS."

There is another type of ordering called **lexicographic ordering** that is generally used in searching words from a dictionaries and indices of books and it is defined by extending a given total ordering $\leq_A$ on the alphabet A to a total ordering $\leq_L$ on A* as follows:

Let a and b be any two strings in A*. Without loss of generality, suppose $|a| \leq |b|$. Let γ be the longest common prefix of a and b that is the longest string such that $\gamma w = a$ and $\gamma z = b$ for some w and z in A*. There are three cases, exactly one of which hold:

1. $W = z = V$ (a and b are identical);
2. $W = V$ and $z = V$ (a is proper prefix of b);
3. $W = x\alpha$, $z = y\beta$, $x \neq y$; $x, y \in A$, and $\alpha, \beta \in A^*$.

The relationship between a and b is defined in each case as follows:

1. $a \leq_L b$ and $b \leq_L a$;
2. $a \leq_L b$ and $b \nleq_L a$;
3. If $x \leq_A y$ then $a \nleq_L b$ and $b \leq_L a$; else $a \nleq_L b$ and $b \nleq_L a$;

THEOREM: Given any finite alphabet A and any total ordering $\leq_A$ on A, the lexicographic ordering $\leq_L$ defined by extending $\leq_A$ is a total ordering on A*.

PROOF:

$\leq_{L}$ is reflexive and antisymmetricfrom the above definition. So now we have to prove that the relation is transitive.

Our main focus is to prove that $x_1 \leq_L x_2$ and $x_2 \leq_L x_3$ implies $x_1 \leq_L x_3$. We have to deal with nine cases, that can be labelled as (i, j) ,

where (i) is the case by which $x_1 \leq_L x_2$ and

(j) is the case by which $x_2 \leq_L x_3$.

Case 1: (1, 1), (1, 2), (1, 3) . In all these cases $x_1 = x_2$. By substituting x_1 for x_2 in $x_2 \leq_L x_3$ we obtain $x_1 \leq_L x_3$ which we are interested to prove.

Case 2: (2, 1) , (3, 1). In these cases $x_2 = x_3$. By substituting x_3 for x_2 in $x_1 \leq_L x_2$ we again obtain $x_1 \leq_L x_3$.

Case 3: (2, 2). In this case $x_2 = x_1 z_1$ and $x_3 = x_2 z_2$. By substituting $x_1 z_1$ for x_2 in $x_3 = x_2 z_2$ we get

$x_3 = x_1 z_1 z_2$ which satisfies case 2 of the definition of $\leq_L$. Thus, $x_1 \leq_L x_3$.

Case 4 : (2, 3) In this case $x_2 = x_1 z = \gamma a \alpha$, $x_3 = \gamma b \beta$ and $a \leq_A b$.

- There are two subcases , depending on whether a falls in x_1 or in z.
- If a falls in x_1 then x_1 is divided into $x_1 = \gamma a \delta$ for some δ, and so case 3 of the definition gives $x_1 \leq_L x_3$.
- If a falls in z, then x_1 is a prefix of x_3 and $x_1 \leq_L x_3$ by case 2 of the definition of $\leq_L$.

Case 5: (3, 2) In this case $x_1 = \gamma a \alpha$, $x_2 = \gamma b \beta$, $a < b$, and $x_3 = x_2 z$. Since $x_3 = \gamma b \beta z$ it follows that $x_1 \leq_L x_3$ by case 3 of the definition.

Case 6: (3, 3) In this case $x_1 = \gamma_1 a_1 \alpha_1$, $x_2 = \gamma_1 b_1 \beta_1 = \gamma_2 a_2 \alpha_2$, $x_3 = \gamma_2 b_2 \beta_2$, $a_1 < b_1$ and $a_2 < b_2$

There are three subcases, depending on whether a_2 falls in γ_1 b_1 and β_1.

- If a_2 occurs as a part of γ_1 , then $x_1 = \gamma_2 a_2 \alpha_3$, and so $x_1 \leq_L x_3$ by case 3.

- If a_2 falls in b_1 then $\gamma_1 = \gamma_2$, $a_1 \leq_A b_1 = a_2 \leq_A b_2$ and so $x_1 \leq_L x_3$ by case 3.
- If a_2 falls in β_1 then , $x_3 = \gamma_1 b_1 \beta_3$ for some β_3 , so that $x_1 \leq_L x_3$ by case 3.

In each case we have shown that $x_1 \leq_L x_2$ and $x_2 \leq_L x_3$ implies $x_1 \leq_L x_3$ so that $\leq_L$ is transitive.
Hence, proved.

A* - The **enumeration ordering** and is denoted as $\leq_E$ --In this strings of unequal length are ordered by length, and strings of equal length are ordered exactly as they are by $\leq_L$. For example, if A = {a, b}, the enumeration of A* is

$\wedge$ α, b, aa, ab, ba, bb, aaa, aab, aba, abb, baa, **…**

Example:

1. Arrange the following strings into ascending order according to the definition of lexicographic ordering

   AND, ANT, AN, BAN, BALL, BAND, CAR, CART

   ➔ AN, AND,ANT,BAN, BALL, BAND, CAR, CART

2. Arrange the above into ascending order according to the definition of the enumeration ordering.

   → AN, AND, ANT, BAN, CAR, BALL, BAND, CART

3. Suppose A has an n elements. How many strings are there in A^k?

   ➔ n^k

4. List all the elements of $\{a, b\}^{3.}$

   ➔ aaa, aab, aba, abb, baa, bab, bba, bbb

5. List all the elements of $\{a, b, c\}^3 \cap \{b, c\}^*$

   ➔ bbb, bbc, bcb, bcc, cbb, cbc, ccb, ccc.

Check Your Progress 4

1. What is Lattice and define **Join-semilattice?**

   ____

   ____

   ____

2. Explain the concept of **lexicographic ordering**

   ____

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   ____

4.10 LET'S SUM UP

It has got wide application in scheduling of system tasks, represent a network of processing elements, helps to represent casual relations between events, used extensively in Genealogy and version history and also used for compact representation of a sequences i.e. data compression.

4.11 KEYWORDS

1. **Catenation-** Arrange in a series of rings or chains
2. The **order** of a **digraph** G is the number of vertices
3. ***Self-loop*** - is an edge that connects a vertex to itself.
4. The ***outdegree*** of a vertex is the number of edges pointing from it.
5. The ***indegree*** of a vertex is the number of edges pointing to it.

4.12 QUESTION FOR REVIEW

1. Draw the digraph for the relation ⊆ on all the nonempty subsets of the set {0, 1, 2}
2. Show that if $x = 2^k$ for some non-negative k and y is odd, then x and y are relatively prime.
3. Prove that the rational numbers are countable.
4. Let A = {1, 2, 3, 4}. Which ordered pairsLet A = {1, 2, 3, 4}. Which ordered pairs are in the relation R = {(a, b) | a < b} ?are in the relation R = {(a, b) | a < b}

4.13 SUGGESTED READINGS

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4.14 ANSWER TO CHECK YOUR PROGRESS

1. Explain the concept of Digraph and example – 4.1 & explain the transitive property with representation– 4.2
2. Explain the concept and one related lemma and example – 4.3
3. State the concept – 4.5.2
4. Explain the concept – 4.4.1
5. Explain the concept – 4.4 & state the condition important for equivalence relation
6. Explain the concept in 4.6.2
7. Explain Poset diagram concept with example --4.6 & and minimal element --4.6.4
8. D
9. Explain the concept with example – 4.9.1 & explain the concept of lattice and join semi lattice-– 4.8

UNIT 5 : RELATIONS AND DIAGRAPHS II

STRUCTURE

5.0 OBJECTIVES

- Different Operation on Relations
- Concept of Paths & Closures
- Directed Graph and Adjacency Matrix
- Boolean Matrices and Inner Product

5.1 OPERATIONS ON RELATIONS:

- There are certain operations on sets that can also be applied on relations and no further concept or definition is required for the same like complement, union, intersection, difference.
- Suppose Ris the proper subset of $A_1 \times \ldots \times A_n$ then

- Complement of R is R = $(A_1 \times \ldots \times A_n)$ - R
- We can also consider the relationships between the usual ordering relations like $\leq$. We can say that $\leq$ is union of the relation < and =; the relation < is the relation obtained from $\leq$ minus =.

5.1.1 N-Ary RELATIONS:

CONCEPT: Let R $\subseteq A_1 \times \ldots \times A_n$ be an **n-ary** relation and lets$_1$, s_2 , … , s_k be a subsequence of the component position 1, **… ,** n of R.
The **projection** of R with respect to s_1, s_2 , … , s_k is the k-ary relation.
$\{(x_1, \ldots, x_k) \mid (x_1, \ldots, x_k) = (a_{s1}, \ldots, a_{sk})$ for some $(a_1, \ldots, a_n) \in R\}$

Example: If R {a, b, c }^{3} is the set of ordered triples {(a,a,a), (a, b ,c), (b, b, c), (a,a, c), (b, a, c),(b, c, c),(a, c, c)}.
The projection of R with respect to the first and third components is the binary relation { (a, a) , (a, c) , (b, c) }.
In this case the projection of R with respect to the first component is the unary relation (i.e. the set) {a, b}.

5.1.2 Join of Relations

CONCEPT: Let $R_1 \subseteq A_1 \times A_2 \times \ldots \times A_n$ and $R_2 \subseteq B_1 \times B_2 \times \ldots \times B_m$ be relations and suppose $A_i = B_j$ for some i = j.
The join of R_1 and R_2 with respect to component i of R_1 and component j of R_2 is the relation

$\{(a_1, \ldots, a_n, b_1, \ldots, b_n) \mid (a_1, \ldots, a_n) \in R_1, (b_1, \ldots, b_n) \in R_2$, and $a_i = b_j\}$

Example: Suppose R_1 = {(a, b), (a, c), (b , a) $\subseteq$ (a, b, c)^2} and R_2 = {(a, b, x), (c, a, y), (a, a, x) , (a, c, x) $\subseteq$(a, b, c)$^2 \times$ (x, y)} .

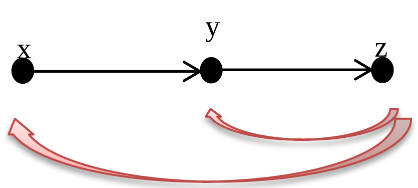
Then the join of R_1 and R_2 with respect to the first component of $R_1$and the second component of R_2 is the relation

{(a, b, c, a, y), (a, b ,a, a, x), (a, c, c, a, y), (a, c, a, a, x), (b, a, a, b, x)}

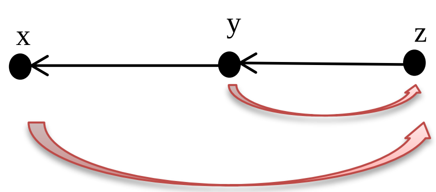
5.1.3 Inverse Of Relations:

CONCEPT: Suppose $R \subseteq A \times B$. the inverse of R, denoted by R^{-1}, is the relation $\{(y, x) \mid (x, y) \in R\}$.

Example: The inverse of the relation R = {(x, y), (y, z), (z, y), (z, x) }is the relation {(y, x), (z, y), (y, z), (x, z)} which can be shown in diagraphs as follows:



R



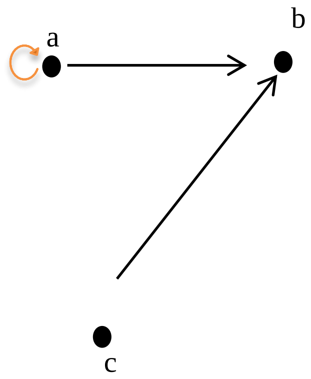
R^{-1}

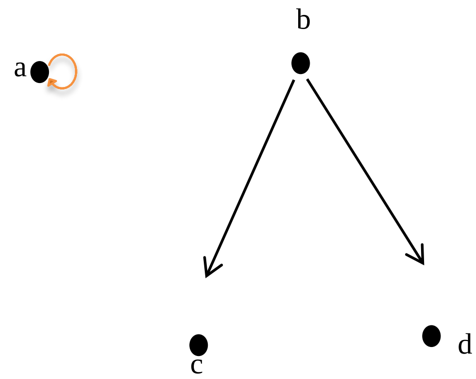
In the above digraphs of the inverse of a relation has exactly the edges of the digraph of the original relation, but the directions of the edges are reversed.

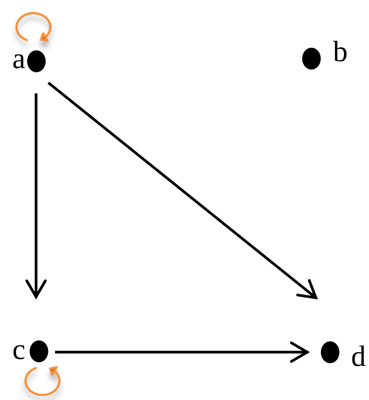
5.1.4 Composition Of Relation:

CONCEPT: Suppose $R_1 \subseteq A \times B$ and $R_2 \subseteq B \times C$. the **composition** of R_1 and R_2 denoted by R_1**.** R_2 is the relation $\{(x, z) \mid (x, y) \in R_1$ and $(y, z) \in R_2\}$

Example: The composition of R_1**.** $R_2$of the relation R_1 = { (a, a), (a, b), (c, b) } and R_2 = { (a, a), (b, c), (b, d) } is the relation {(a, a), (a, c) (a, d), (c, c), (c, d)}. These relation are shown below:







[NOTE: Composition of relation is associative, That is if R, S, T are relations then (RS) T = R (ST)

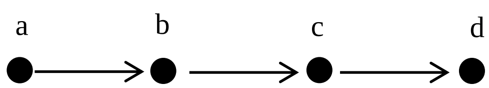
The notation R^k is used for the iterated composition of R with itself which means $R^1 = R$, and $R^{k+1} = R^k . R$ for $k \geq 1$.]

5.1.5 Transitive Closure

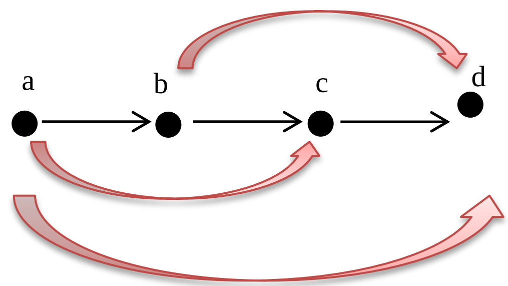
CONCEPT: Suppose $R \subseteq A \times A$. The transitive closure of R, denoted by R^+ is $R \cup R^2 \cup R^3 \ldots = \cup_{k \geq 1} R^k$.

The Transitive reflexive closure of R denoted by R^*, is $R^+ \cup \{(a, a) \mid a \in A\}$.

Example: The transitive closure R^+ of the relation R = {(a, b), (b, c), (c, d)} is the relation {(a, b) , (a, c), (a, d), (b, c), (b, d), (c, d)}.This is shown below in the digraphs:



R



R^+

THEOREM: R^+ is the smallest relation containing R that is transitive.

PROOF:

- First we will prove that R^+ is transitive.
- Suppose x R^+ y and y R^+ z. Then, since $R^+ = \cup_{k \geq 1} R^k$, x R^i y and y R^j z for some i, j ≥ 1. Thus, x R^{i+j} z and so x R^+ z.
- Suppose R ⊆Q and **Q** is transitive. Suppose R^k⊆Q. Then, since R ⊆Q and **Q** is transitive, R ⊆ Q. Thus, by induction on k, R ⊆Q for every k ≥ 1, and so R ⊆Q.
- Similarly, the transitive reflexive closure of a binary relation is the smallest relation that contains it and is both transitive and reflexive.
- For example, the symmetry property says that if **R** includes a pair (x, y) then **R** must also include (y, x).
- The symmetric closure of a relation **R** is thus the set R U R^{-1}, which is the smallest symmetric relation that includes **R**.
- In general if **P** is a property such that **P** can be made true for any set by adding certain elements to the set, we can call **P** a closure

property and define the P-closure of a set to be the smallest set that contains it and satisfies property **P**.

[NOTE: $\sum^{+}$ - the set of all non-null character strings over alphabet $\sum$, is another example of such a closure.
If x and y are elements of $\sum^{+}$ then xy is an element of $\sum^{+}$.]

Example:

1 . Let P ={ (x, y, x **.**y) | x and y are integers} and Q = {(x,x,z) | x and z are integers}.

a. What is P ∩ Q?

{(x, x, x^2) | x is an integer}.

b. What is the projection of P ∩ Q with respect to the first and third components?

{(x, x^2) | x is an integer}.

c. Let R be the join of P and {1, 3} with respect to the first component of P. Describe R

{(1, y, 1 **.** y, 1) | y is an integers} U {(3, y, 3 **.** y, 3) | y is an integer}

d. Let T be the join of R and {5} with respect to the second component of R. Describe T.

{(1, 5, 5, 1, 5), (3, 5, 15, 3, 5)}

e. What is the projection of T with respect to the third component?

{ 5, 15}

2. Let R = {(x, y) | x = y**.** z for some z greater than one, and x, y, z are positive integers}

   a. What is R ∩ R^{-1} ? Is R symmetric? Reflexive?

      The intersection is empty. The relation R is not symmetric, nor is it reflexive.

b. Prove that R ⊆R . R. What is the transitive closure of R?

If x = y. a and y = z .b then x = z . (b . a). If a and b are greater than one , then so is b . a. The relation R is its own transitive closure, since it is already transitive.

c. What is the projection of R with respect to the first component?

The projection of R with respect to the first component is the set of integers ≥

Check Your Progress 1

1. What is join of Relation?

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2. Explain Transitive Closure.

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5.2 PATHS AND CLOSURES

CONCEPT:

- A **directed path** in a digraph A = (V, E) is a sequence of zero or more edges e_1, **…**, e_n in E such that for each $2 \leq i \leq n$, e_{i-1} is to the vertex that is, e_i may be written as (v_{i-1}, v_i) for each $1 \leq i \leq n$.

- Such a path is said to be from v_0 to v_n, and **its length is n**.

- In this case v_0 and to v_n are called **endpoints** of the path.

- A **non-directed path** in G is a sequence of zero or more edges e_1, **…**, e_n in E for which there is a sequence of vertices v_0, **…**, v_n such that $e_i = (v_{i-1}, v_i)$ or $e_i = (v_i, v_{i-1})$ for each $1 \leq i \leq n$.

- A path is **simple** if all edges and vertices on the path are distinct, except that v_0 and v_n may be equal.

- A path of length ≥ 1 with no repeated edges and whose endpoints are equal is a **circuit.**

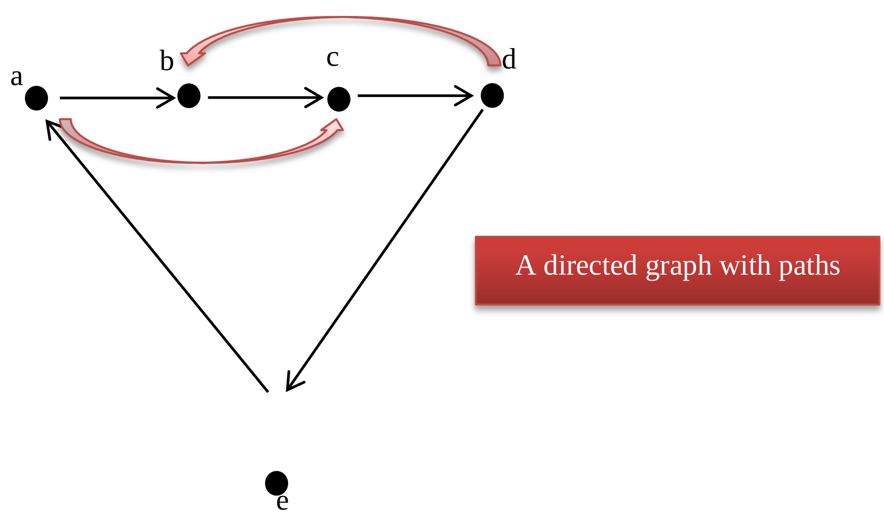
- A simple circuit is called a **cycle.**

[NOTE: The definition of Simple, circuit and cycle apply equally to directed and non-directed paths

A path of length zero is permitted, but it does not have a unique pair of endpoints. Such a path has no edges and can be viewed as being from a vertex to itself.]

- A path e_1, **…**, e_n is said to **transverse** a vertex x if one (or more) of the e_i is to or from x and x is not serving as one of the endpoints of the path or more precisely, if $e_i = (x, y)$, then $2 \leq i \leq n$, or if $e_i = (y, x)$, then $1 \leq i \leq n - 1$.

Example:



From the above figure we can observe:

- There are two simple directed paths from a to d which are (a, b), (b , c), (c, d), and (a, c), (c, d).
- Simple non directed paths from a to d:- (a, b) and (b, d).
- Non trivial directed cycles:- (a, b), (b, c), (c, d), (d, e), (e, a)
- Non directed cycles which includes all directed cycles + (a, b), (b, c), (c, a)

THEOREM: If A =(V, E) is a digraph, then for $n \geq 1$, $(x, y) \epsilon E^n$ if and only if there is a directed path of length n from x toy in A.

PROOF:

For n = 1, $E^n = E$. The definition of the path guarantees that $(x, y) \epsilon E$ if and only if there is a path 1 from x to y, since a path of length 1 is an edge of A. For n > 1, we assume the theorem is true for n – 1, and divide the proof into two parts:

I. Suppose (v_0, v_1) , **…**, (v_{n-1}, v_n) is a directed path from v_0 to v_n. Then $(v_0, v_{n-1}) \epsilon E^{n-1}$, by induction.
By definition $E^n = E^{n-1}$ **.** E, and **,** and$(v_{n-1}, v_n) \epsilon E$, so that $(v_0, v_n) \epsilon E^n$.

II. Suppose $(v_0, v_n) \epsilon E^n$. Then since $E^n = E^{n-1}$ **.** E, there exist some v_{n-1} such that $(v_0, v_{n-1}) \epsilon E^{n-1}$ and$(v_{n-1}, v_n) \epsilon E$.
By the inductive hypothesis, there is directed path (v_0, v_1) , **…**, (v_{n-2}, v_{n-1}) of length n – 1 from v_0 tov_{n-1}. Adding (v_{n-1}, v_n) to this path gives the path of length n desired.

ADD-ONS: If there is non-trivial directed path from a vertex **x** to a vertex **y** there must be an edge directly from x to y.
If A = (V, E) is a digraph then E is transitive if and only if every directed path in A has a “short-cut”.

COROLLARY: If A = (V, E) is a digraph then for any two vertices x and y in V, (x, y) ϵ E^+ if and only if there is nontrivial directed path from x to y in A.

PROOF: From previous theorem, we know that if there is a directed path from x to y in A of some length $n \geq 1$, then (x, y) ϵ E^n, so that (x, y) ϵ E^+ .

Conversely, if (x, y) ϵ E^+ , then (x, y) ϵ E^n, for some $n \geq 1$ and thus we can say that there is a directed path from x to y of length n.

Example: Let A ={ a, b, c, d, e} and let R = { (a, a), (a, b), (b, c), (c, d), (c, e) ,(d, e)}.

From the definition of transitive closure since, $R^{+} U_{k \geq 1} R^k$ we need to compute R^k for each k, and take their union. By the definition of composition, we get

R^2 = { (a, a), (a, b),(a, c), (b, e), (b, d), (c, e)}

R^3 = { (a, a), (a, b),(a, c), (a, d), (a, e), (b, e)}

R^4 = { (a, a), (a, b),(a, c), (a, d), (a, e)}

R^5 = { (a, a), (a, b),(a, c), (a, d), (a, e)}

We can see from above that $R^4 = R^5$ so it will follow that $R^4 = R^5 = R^6 = R^5$ for all $k \geq 4$.

Thus, R^k is just the union of these sets, so that

R^k = {(a, a), (a, b),(a, c), (a, d), (a, e), (b, c) , (b, d), (b, e),(c, d),(c, e), (d, e)}.

CONCEPT:

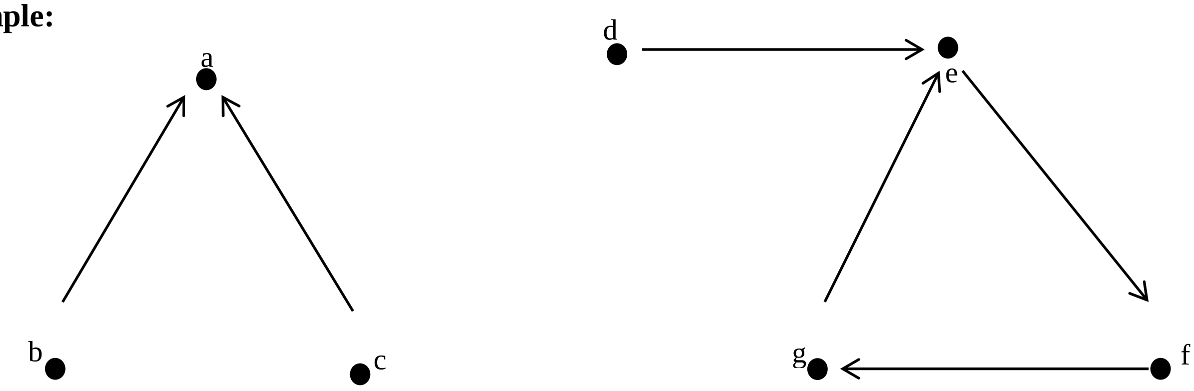
- A pair of vertices in a digraph is **weakly connected** if there is a non- directed path between them.

- ➢ They are **unilaterally connected** if there is a directed path between them.

- ➢ They are **strongly connected** if there is a directed path from x to y and a directed path from y to x.

- ➢ A **subgraph** of A^1 of a graph A is a(weakly, unilaterally, strongly) connected component if it is a maximal (weakly, unilaterally, strongly) connected subgraph.

Example:



A graph illustrating connectivities

- ✓ The above graph comprises of vertices { a,b,c, d,e,f, g} and their incident edges, is not even weakly connected.
- ✓ The subgraph comprising of vertices {a, b, c} and the edges {(a, b), (c, b)} is weakly connected, but not connected by either of the stronger definitions, since there is no directed path between a and c.
- ✓ The subgraph consists of vertices {d, e, f, g} and their incident edges is unilaterally connected, but not strongly connected, as there is no directed path from e to d.
- ✓ {e, f, g} – vertices of nontrivial strongly connected subgraph & its edges are {(g, e),(e, f), (f, g)}.
- ✓ Two subgraph with a vertex sets {a, b, c} and {d, e, f, g} has two weakly connected components.

- ✓ Individual vertices – a, b , c and d and the subgraph {(e, f, g)} ,{(e, f), (f, g), (g, e)} – strongly connected components.
- ✓ Unilateral connected components – {(a, b), (b, a)} and {(b, c), (c, b)}.

5.3 DIRECTED GRAPH AND ADJACENCY MATRICES:

CONCEPT: Let S be any set and m, n be any positive integers. An m × n matrix over S is a two dimensional rectangular array elements with m rows and n columns.

The elements are doubly indexed with the first index indicating the row number and the second indicating the column number as illustrated below:

$$
A = \begin{array}{c c}
 & \begin{array}{cccc} \text{Column 1} & \text{Column 2} & \ldots & \text{Column n} \end{array} \\
\begin{array}{c} \text{Row 1} \\ \text{Row 2} \\ \cdot \\ \cdot \\ \cdot \\ \text{Row m} \end{array} &
\begin{pmatrix} A(1, 1) & A(1, 2) & \ldots & A(1, n) \\ A(2, 1) & A(2, 2) & \ldots & A(2, n) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ A(m, 1) & A(m, 2) & \ldots & A(m, n) \end{pmatrix}
\end{array}
$$

Boolean Matrices- These are the matrices over the set {0, 1}. There is a natural one –to-one correspondence between the binary relations and the square Boolean matrices.

CONCEPT: Let E be any binary relation on a finite set $V = \{v_1, \ldots, v_n\}$. The adjacency matrix of E is the n × n Boolean matrix A defined by A(i, j) = 1 if and only if $(v_i, v_j) \in E$.

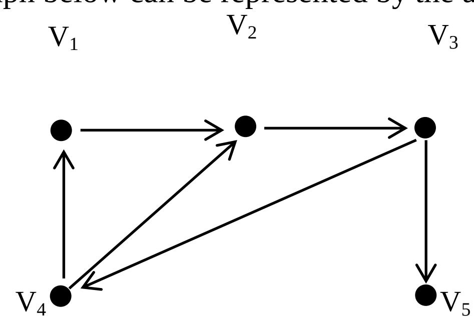
NOTE: the interpretation of adjacency matrices depends on the presumed ordering of the set V.

Example: The relation ≤ on the set {0,1, 2, 3, 4} is represented by the adjacency matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

All diagonal entries in the above matrix is 1. This is because the relation ≤ is reflexive and it is true for every reflexive relation.

Example: The digraph below can be represented by the adjacency matrix as follows:



$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

CONCEPT: Let S be any set and let $\oplus$ and $\otimes$ be any two binary operators defined on the elements of S. Assume that $\oplus$ is associative. The inner product of $\oplus$ and $\otimes$, denoted by $\oplus.\otimes$ is defined for n × n matrices over S by A $\oplus.\otimes$B = D such that

$$D(i, j) = [A(i, 1) \otimes B(1, j)] \oplus \ldots \oplus [A(i, n) \otimes B(n, j)]$$

We can write $(\oplus.\otimes)^k A$ to denote the matrix A in the case that k = 1 and for k > 1 to denote the matrix $[(\oplus.\otimes)^{k-1} A \oplus.\otimes A]$.

For any single scalar operator $\oplus$ we will also write $A \oplus B$ for matrices A and B to denote the matrix E such that

$$E(i, j) = A(i,j) \oplus B(i, j)$$

NOTE: In the above case the inner product is the usual definition of matrix product.

Example: Let S be the set {0, 1} and let $\oplus$ and $\otimes$ be the operators OR and AND defined by the table:

x	**y**	**x OR y**	**x AND y**
0	0	0	0
0	1	1	0
1	0	1	0
1	1	1	1

Let A and B be the matrix in the previous two above example of Adjacency matrix.

The inner product A OR **.** AND B is the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We will clarify how we had obtain these elements of the matrix.

- ✓ Consider entry in row 2 and column 2 - it is obtained from the second row of A and second column of B, and is (0 AND 1) OR (1 AND 0) OR (1 AND 0) OR (1 AND 1) OR (1 AND 0) = 1.

- ✓ The entry in row 2 and column 3 is obtained from the second row of A and the third column of B, (0 AND 1) OR (0 AND 1) OR (1 AND 1) OR(0 AND 0) OR (0 AND 0) = 1.

CONCEPT: When the two operations $\oplus$ and $\otimes$are the particular operations of OR and AND respectively, then we can refer to the inner product as **Boolean Product.**

THEOREM: Let R_A and R_B be binary relation on a set V = $\{v_1 \ldots v_n\}$, represented by adjacency matrices A and B respectively. Then the Boolean Product A OR **.** AND B is the adjacency matrix of the relation$R_A^n . R_B$ and the matrix (OR **.** AND)^n A is the adjacency matrix of the relation R_A**.** Here (OR **.** AND)^{2}A means A OR **.** AND A.

PROOF:

$R_A . R_B$ = Rc where Rc = { (v_i , v_k) | $(v_i , v_j) \in R_A$ and $(v_j , v_k) \in R_B$ for some j}. Thus if C is adjacency matrix of Rc then C (i, k) = 1 if and only if for some j, A (v_i , v_j) = 1 and A (v_j , v_k) = 1.

It is same like if we say C (i, k) =[A(i, 1) AND B (1, j)] OR **…**OR [A(i, n)AND B (n, j)] which is (A OR **.** AND B) (i, j).

The theorem is the direct consequence of the definitions.

COROLLARY: Let A be the adjacency matrix of any binary relation R on a set V = $\{v_1 , \ldots, v_n\}$. Then the adjacency matrix of the transitive closure R^+ is given by A OR (OR **.** AND)^2 OR **…** OR (OR **.**AND)^n A.

PROOF: It follows from the above theorem and $R^+ = R \cup R^2 \cup \ldots \cup R^n$.

THEOREM:

Suppose G = (V, E) is a directed graph and A is its adjacency matrix. Let $\oplus$ and $\otimes$ denote the operations.

$$x \oplus y = \begin{cases} x & if\ x > y \\ y & otherwise \end{cases}$$

$$x \otimes y = \begin{cases} x + y & if\ x > y\ and\ y > 0 \\ 0 & otherwise \end{cases}$$

$$L^k = \begin{cases} A & for\ k = 1 \\ L^{k-1} \oplus (L^{k-1} \oplus . \otimes A) & for\ k > 1 \end{cases}$$

Then $L^k(i, j)$ is the length of the longest non trivial directed path from v_i to v_j that has length $\le k$, unless $L^k(i, j) = 0$, in which case no such path exists.

PROOF: The proof is by induction on k. For k = 1 we have $L^k = A$. Since A (i, j) =1 if and only if there is a directed path of length 1 from v_i to v_j, the conclusion follows directly from the definition of adjacency matrix.
For k > 1 we can assume the theorem holds for smaller values of k. $L^k(i, j) = L^{k-1}(i, k) \oplus [L^{k-1}(i, 1) \otimes A(1, j)] \oplus \ldots \oplus [L^{k-1}(i, n) \otimes A(n, j)]$.

In other words $L^k(i, j)$ is the maximum of $L^k(i, j)$ and all of $L^{k-1}(i, t) \otimes A(t, j)$ for $1 \le t \le n$.

We will consider two cases.
Case 1: Suppose there is nontrivial directed path from v_i to v_j of length $\le k$.

- We will prove that $L^{k-1}(i, j)$ and $L^{k-1}(i, t) \otimes A(t, j)$ are zero in this case.
- Let $L^{k-1}(i, j)$ is nonzero, then by induction there is nontrivial path of length $\le k - 1$ from v_i to v_j

- If $L^{k-1}(i, t) \otimes A(t, j)$ is nonzero for some t , then by induction there is directed path from v_i to v_j and an edge from v_i to v_j that can be combined to form a directed path of length $\leq k$ from v_i to v_j.

Case 2: Suppose there is nontrivial directed path from v_i to v_j of length $\leq k$.

- We will choose one path of maximum length. Let *l* be the length of this path and (v_i, v_j) be the last edge on it.

- By definition of adjacency matrix, A(t, j) = 1. We will consider subcases whether *l* is greater than 1.

- If *l* = 1, since we are considering that k > 1, and as we consider *l* as length of the longest path of length $\leq k - 1$ and so by induction $L^{k-1}(i, j) = 1$.

- It implies $L^{k-1}(i, j)$ is at least 1.

- But it can happen one of the $L^{k-1}(i, t)$ A (t, j) is greater.

- Suppose that $L^{k-1}(i, t) \otimes A(t, j)$ is non zero

- So from case 1there must be directed path from v_i to v_j of length > 1 and $\leq k$ which would be longer than *l* that is a contradiction.

- Thus, $L^{k-1}(i, j) = 1$.

- If *l*> 1, there is nontrivial directed path from v_i to v_t of length *l* – 1.

- But there cannot be any directed path from v_i to v_t longer than $l - 1$ and shorter than k, if we try to figure out some path longer than l than it is a contradiction to the definition of l.

- Thus by induction $L^{k-1}(i, t) = l - 1$, so we have

  $L^{k-1}(i, t) \otimes A(t, j) = l$ Thus $L^{k}(i, j)$ is at least l.

- Suppose $L^{k}(i, j)$ is greater than l which leads to $L^{k-1}(i, j)$ is greater than l or some term $L^{k-1}(i, t) \otimes A(t, j) > l$ which again contradicts the definition.

- If $L^{k-1}(i, t) \otimes A(t, j) > l$, $L^{k-1}(i, t) > l - 1$ so by induction there must be a path of length $L^{k-1}(i, t)$ from v_i to v_j and this must not be longer than $k - 1$.

- Such a path could be joined with (v_i, v_j) to obtain a path greater than l and $\leq$ k from which is a contradiction.

- We proved that $L^{k}(i, j) = l$.

5.4 WARSHALL'S ALGORITHM:

Computing the Transitive Closure:

Input: The adjacency matrix of a digraph (V, E)

Output: A new adjacency matrix M, which is the adjacency matrix of (V, E^{+})

Method: For each k from 1 up to n

For each pair (i, j) such that $1 \leq i, j \leq n$ do the following

(*) If M(i, k) = 1, M(k, j) = 1, and M(i, j)=0 then change M(i, j) to 1.

The below figure represents the basic idea of Warshall's algorithm.

Observation:

From figure, there is a path from vertex v_i to v_j that traverses only vertices in $\{v_1, ... v_k\}$. There are two possibilities one is that the path may traverse only vertices in $\{v_1, ... v_{k-1}\}$ or;it traverses only vertices in $\{v_1, ... v_{k-1}\}$ to get to v_k for the first time, may visit v_k several more times via subpaths that traverse only vertices in $\{v_1, ... v_{k-1}\}$ and finally reaches v_j via a subpath that traverses only vertices in $\{v_1, ... v_{k-1}\}$.

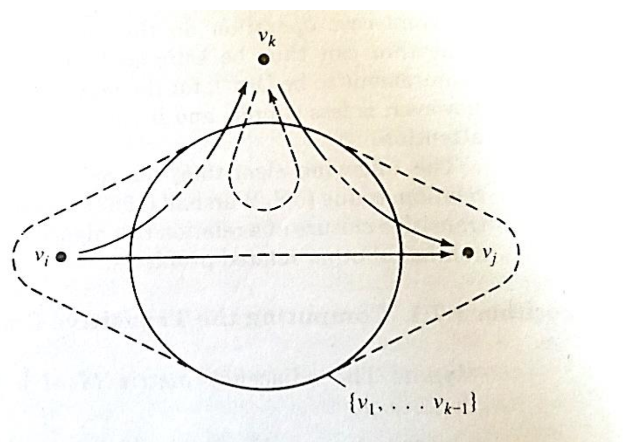
Other way round, for each vertex v_k and all paths through it the main loop of the algorithm considers all of the possible two –step paths from v_i to v_j that go into and out of v_k. The algorithm will build a bypass (v_i, v_j) unless no such bypass already exist.

At last each vertex is bypassed which means that for each path in the original graph there will be direct connection (edge) in the result.

Let $\mathbf{M_0}$ denote the initial matrix **M**. Let $\mathbf{M_k}$ denote the matrix at the end of the $\boldsymbol{k^{th}}$ stage. Note no entry of M is ever set to 0 so that $M_k(i, j) = 1$ implies that$M_{k+1}(i, j) = 1$.

Basically we want to prove that $\mathbf{M_n}$ is adjacency matrix of (V, E^+) and for this, refer the following two theorems.

Warshall's Algorithm



THEOREM 1: $M_k(i, j) = 1$ if there is a nontrivial directed path from v_i to v_j that traverses only vertices in $\{v_1, \ldots, v_k\}$

To prove this theorem we need a lemma.

LEMMA: If there is a directed path in a diagraph from vertex ***a*** to vertex ***b*** and **S** is the set of vertices traversed by this path, then from any vertex c in S there exist directed paths from ***a*** to ***c*** and from ***c*** to ***b*** such that each traverses only vertices in **S– {*c*}**

Proof:

- Suppose **S** is non-empty and ***c*** is an element of **S**.If $S = \emptyset$, the lemma is trivially true, since there is no vertex ***c*** in**S**.
- Let $e_1, \ldots, e_k$ be a directed path from ***a*** to ***b*** that traverses exactly the vertices in **S**.
- By the definition of traverse, there is at least one edge $\mathbf{e_i}$ in the path that is incident to ***c***.
- Let $\mathbf{e}_{\boldsymbol{k}}$be the first edge on the path that is incident to c and let e_{j} be that last edge that is incident from c.
- We are aware that from definition of path $e_i = (x, c)$ and $e_j = (c, y)$ for some ***x*** and ***y.***
- So, $e_1, \ldots, e_i$ is a path from a to c and $e_j, \ldots, e_k$ is a path from c to b and neither of these traverses c.
- Both paths are sub paths of $e_1, \ldots, e_n$ so they traverses only vertices in **S– {*c*}**

Proof of Theorem: We are using induction method on ***k*** for proof.

When k = 0 then $M_0(i, j) = 1$ if $\left(v_i, v_j\right)$ is in E.

Assume the theorem is true for smaller value of k. Let there is a nontrivial directed path from $v_i$to v_j that traverse vertices only in $\{v_i, \ldots, v_k\}$

From the above lemma there is a possibility of either a nontrivial directed path from $v_i$to v_j that traverses only vertices in $\{v_1, \ldots, v_{k-1}\}$ or there is a nontrivial paths from $v_i$to v_k and from $v_k$to v_j using only vertices in $\{v_1, \ldots, v_{k-1}\}$.

In the first case by the inductive hypothesis, $M_{k-1}(i, j) = 1$ and in second case $M_k(i, j)$ is set to 1 by the algorithm.

If we put k =n we can have an immediate corollary.

COROLLARY: if there is non-trivial directed path from $v_i$to v_j in (V, E), then $M_n(i,j) = 1$ where n = |V|

THEOREM 2: If $M(i,j) = 1$ at any time during the execution of the algorithm, there is a non-trivial directed path from $v_i$to $v_j$in (V, E).

NOTE: The step (*) is performed repeatedly, for different values of ***i, j*** and ***k*** in some order. Time '***t***' means that step (*) is performed ***t*** times, time 0 is before the step (*) has been performed at all and time 1 is just after it has been performed once and so on.

PROOF:

Assume $M(i,j) = 1$.We will use induction on time '***t***' at which entry $M \quad (i,j)$ is first set to 1.

The algorithm never changes $M \quad (i,j)$ to 0 that happens at most once for each pair (i, j).

For t = 0, we have original adjacency matrix, if $M(i,j) = 1$ at time t = 0 so (v_i, v_j) is an edge in E is true.

For t > 0, at time t entry $M(i,j)$ is changed from 0 to 1. Then at time t – 1 then for the current value of k $M(i,k) = 1$ and $M(k,j) = 1$ is true.

There must be a non-trivial directed paths from $v_i$to v_k and from $v_k$to v_j by induction and thus joining these paths at $v_k$we will have a nontrivial directed path from $v_i$to v_j.

Application of Warshall Algorithm:

We will first consider the adjacency matrix of relation **R** i.e. $\mathbf{M_0}$ and then construct successively other matrices like $M_1, M_2, \ldots, M_n$ where **n** is the number of vertices for the relation **R**.

Condition (*) is very useful in constructing the $\mathbf{M_k}$ in terms of previously constructed $\mathbf{M_{k-1}}$for each k ≥ 0 i.e. we can obtain $\mathbf{M_k}$(i, j) the **(i, j)** entry of $\mathbf{M_k}$, from certain entries of $\mathbf{M_{k-1}}$**.** For instance, if $\mathbf{M_{k-1}}$(i, j) = 1 then $\mathbf{M_k}$(i, j) = 1 also which means every entry of $\mathbf{M_{k-1}}$is a 1 remains a 1 in $\mathbf{M_k}$**.** If $\mathbf{M_{k-}}$

$_1$(i, j) = 0, then we get a new 1 in a position (i, j) of $\mathbf{M_k}$ only if there were ones in a positions (i, k) and (k, j) of $\mathbf{M_{k-1}}$.

In other way, $\mathbf{M_k}$(i, j) = 1 if $\mathbf{M_{k-1}}$(i, k) = 1 and $\mathbf{M_{k-1}}$(k, j) = 1. Thus, if $\mathbf{M_{k-1}}$(i, j) = 0, so column **k** and row **k** of $\mathbf{M_{k-1}}$ need to be examined and if there is a **1** in position **i** of column **k** and **1** in position **j** of row **k,** a 1 will be entered in position (i, j) of $\mathbf{M_k}$.

The following expression describes it following

$$M_k(i,j) = M_{k-1}(i,j) \vee (M_{k-1}(i,k) \wedge M_{k-1}(k,j))$$

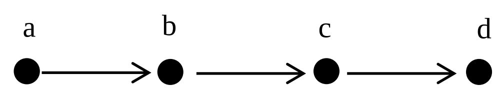
Thus, we may construct $\mathbf{M_k}$ from $\mathbf{M_{k-1}}$ by employing the following procedure.

Step 1: First transfer all 1's of $\mathbf{M_{k-1}}$ to $\mathbf{M_k}$

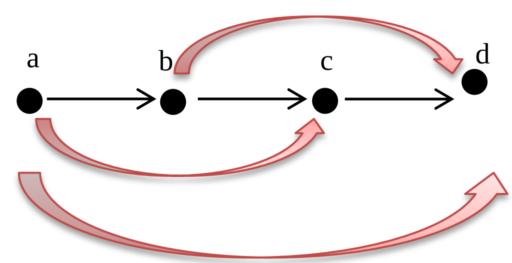
Step 2: In column k of $\mathbf{M_{k-1}}$ where entry is 1 record all the positions $p_1, p_2, \ldots$ and in row k of $\mathbf{M_{k-1}}$ where entry is 1 record all the positions $q_1, q_2, \ldots$

Step 3: Put a 1 in each position (p_s, q_t) of $\mathbf{M_k}$

Example: Using Warshall's algorithm compute the adjacency matrix of the transitive closure (V, E^+) of digraph (V, E).



R



R^+

Solution: First we let

$$M_0 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now we will find M_1. We observe M_0 has 1's in position 1 of column 1 and positions 1 and 2 of row 1. Thus, $M_1(1,1) = 1 = M_1(1,2)$, but since the (1, 1) and (1, 2) entries of M_0 were already transferred to M_1, we introduce no new 1's. Thus, $M_1 = M_0$.

Now, we will compute M_2 so that in this computation we let k = 2. In column of M_1, there is a 1 in position 1 and there is a 1 in a position 3 of row 2. Thus, $M_2(1,3) = 1$. This is only new 1 to be added to M_1.

Hence,

$$M_2 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In similar way we will proceed to compute M_3. We observe that position 1 and 2 of column 3 have 1's, while position 4 and 5 of row 3 have 1's. Thus, $M_1(1,4) = M_3(1,5) = M_3(2,4) = M_3(2,5) = 1$. Therefore,

$$M_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Next observe that M_3 has a 1 in positions 1, 2 and 3 of column 4 while row 4 has a 1 in position 5. Thus, $1 = M_4(1,5) = M_4(2,5) = M_4(3,5)$. None of these require changes. Therefore,$M_3 = M_4$.

Finally, M_4 has a several positions of column 5 but no 1's in row 5. Thus, no new 1's need to be added to M_4. Therefore, $M_5 = M_4 = M_3$ is the adjacency matrix of R^+.

COROLLARY: Warshall's algorithm computes the adjacency matrix of the transitive closure (V, E^+) of digraph (V, E).

Check Your Progress 2

1. What is Boolean Product?

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2. What is Warshall Algorithm?

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5.5 LET'S SUM UP

It has got wide application in scheduling of system tasks, represent a network of processing elements, helps to represent casual relations between events, used extensively in Genealogy and version history and also used for compact representation of a sequences i.e. data compression.

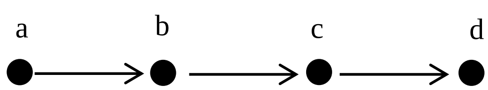
5.6 KEYWORDS

1. A *directed cycle* is a directed path (with at least one edge) whose first and last vertices are the same.
2. The *length* of a path or a cycle is its number of edges.
3. A *directed acyclic graph* (or DAG) is a digraph with no directed cycles

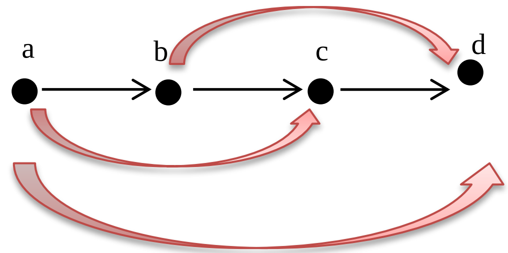
5.7 QUESTION FOR REVIEW

1. What is the longest length possible for a simple directed path in a digraph with ***n*** vertices? How about the longest cycle?

2. Using Warshall's slgorithm, compute the adjacency matrix of the transitive closure of the digraph in the following figure.



R



R^+

3. Let $A=\{\ 0,1,2,3\}$ and a relation R on A be given by

   4. $R=\{\ (0,0), (0,1), (0,3), (1,0), (1,1), (2,2), (3,0), (3,3)\ \}$.

   Is R reflexive? symmetric? transitive?

4. Let m,n and d be integers with $d \neq 0$. Then if d divides $(m\text{-}n)$, denoted by $d \mid (m\text{-}n)$, i.e. $m\text{-}n=dk$ for some integer k, then we say ***m* is congruent to *n* modulo *d***, written simply as $m \equiv n \pmod d$. Let R be the relation of congruence modulo 3 on the set $\mathbb{Z}$ of all integers, i.e.

$$m\ R\ n \Leftrightarrow m \equiv n \text{ (mod) } 3 \Leftrightarrow 3 \mid (m\text{-}n)\ .$$

Show R is an equivalence relation.

5.8 SUGGESTED READINGS

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5.9 ANSWER TO CHECK YOUR PROGRESS

1. Explain the concept with example - 5.1.2
2. Explain the concept with example - 5.1.5
3. State the concept and explain the theorem with proof---5.3.4
4. State the concept and explain the theorem with proof---5.4

UNIT 6: RECURRENCE RELATION

STRUCTURE

6.0 OBJECTIVE

- What is a Recurrence Relation and its application in solving Tower of Hanoi Problem?
- Homogeneous and Non-Homogeneous Recurrence Relation
- Application of recurrence relation in finding a general term of sequence
- Concept of Difference Table and Line in a plane in general position

6.1 INTRODUCTION:

Recursion— breaking an object down into smaller objects of the same type— is a common approach in mathematics and computer science. Like for instance, in an induction proof we establish the truth of a statement P (n) from the truth of the statement P (n − 1). In computer programming, a recursive algorithm solves a problem by applying itself to smaller instances of the problem and on the mathematical side, a recurrence equation describes the value of a function in terms of its value for smaller arguments.

6.2 WHAT IS A RECURRENCE?

A **recurrence relation** for a sequence $a_0, a_1, a_2 \ldots$ is a formula (equation) that relates each term a_n to certain of its predecessors $a_0, a_1 \ldots, a_{n-1}$.
The **initial conditions** for such a recurrence relation specify the values of $a_0, a_1, a_2 \ldots, a_{n-1}$.

Let us consider the following example, the recursive formula for the sequence 3, 8, 13, 18, 23
is $a_1 = 3$, $a_n = a_{n-1} + 5$, $2 \leq n < \infty$.
Here, $a_1 = 3$ is the initial condition.

Similarly, consider the infinite sequence as 3, 7, 11, 15, 19, 23 … which can be defined by the following recursive formula as

$a_1 = 3$, $a_n = a_{n-1} + 4$, $2 \leq n < \infty$.

Example: Find the sequence represented by the recursive formula $a_1 = 5$, $a_n = 2a_{n-1}$, $2 \leq n \leq 6$.

Solution: The initial condition is $a_1 = 5$ and n satisfies the condition $2 \leq n \leq 6$. Thus,

$a_2 = 2a_1 = 10$,

$a_3 = 2a_2 = 20$,

$a_4 = 2a_3 = 40$,

$a_5 = 2a_4 = 80$,

$a_6 = 2a_5 = 160$.

Hence the given recurrence formula defines the finite sequence

5, 10, 20, 40, 80, 160.

6.3 THE TOWERS OF HANOI

In the Towers of Hanoi problem, there are three posts and seven disks of different sizes. Each disk has a hole in the center so that it fits on a post. At the start, all seven disks are on post #1 as shown below. The disks are arranged by size so that the largest is on the bottom and the smallest is on top. The goal is to end up with all seven disks in the same order on a different post. Because of two restrictions, this is not trivial. First, the only permitted action is removing the top disk from a post and dropping it onto another post. Second, a larger disk can never lie above a smaller disk on any post. (Note: We cannot move whole stack of disks at once and then drop them all on another post!)

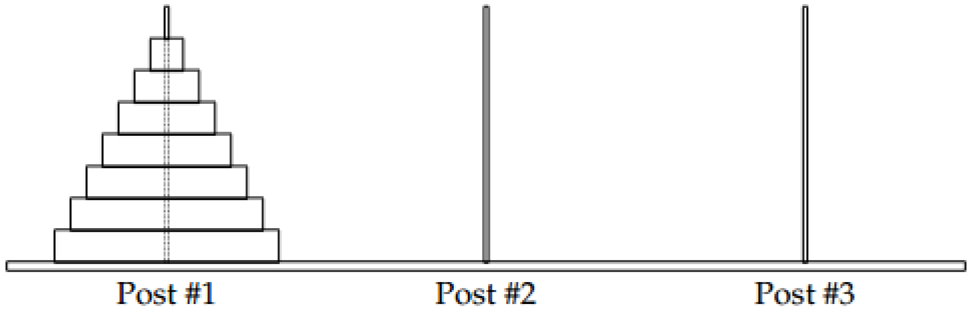


Fig 4.1: Tower of Hanoi Problem

One approach to this problem is to consider a simpler variant with only three disks. We can quickly exhaust the possibilities of this simpler puzzle and find a 7 move solution such as the one shown below. (The disks on each post are indicated by the numbers immediately to the right. Larger numbers correspond to larger disks.)

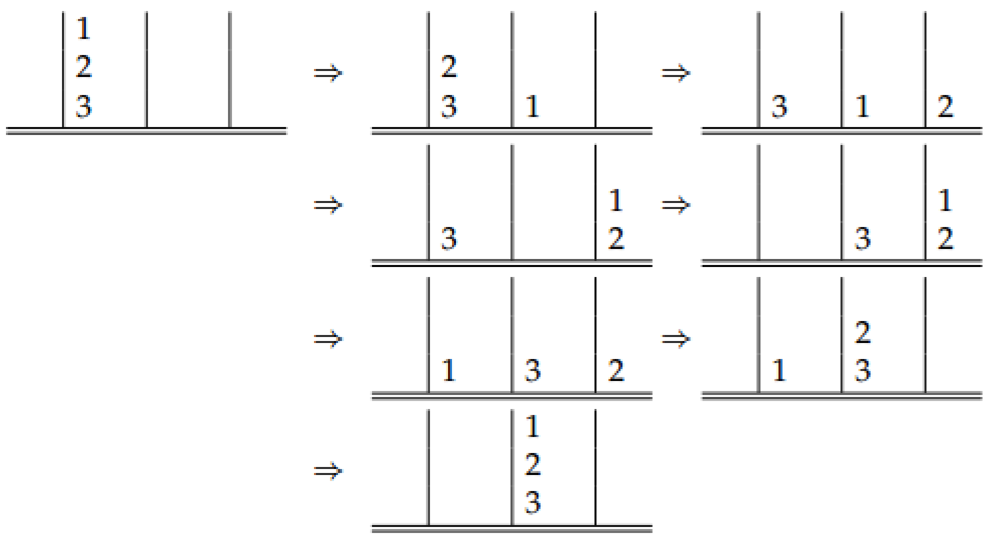
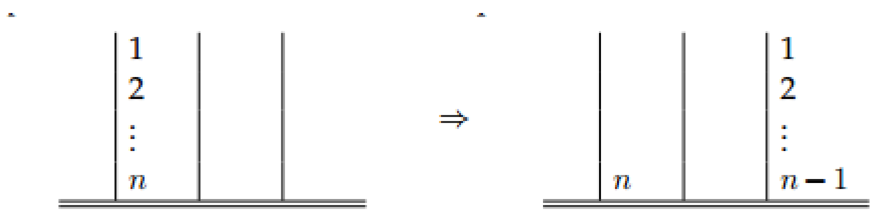


Fig 4.2: Solution for the Tower of Hanoi Problem when 3 disks are considered

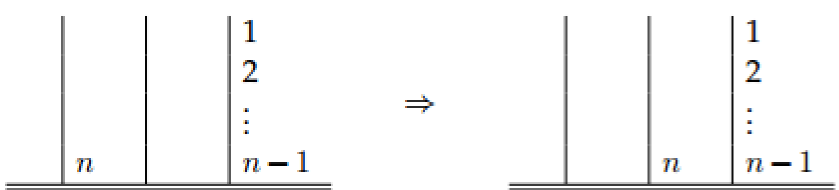
Finding a Recurrence

The Towers of Hanoi problem can be solved recursively as follows. Let T_n be the minimum number of steps needed to move an **n** disk tower from one post to another. For example, a bit of experimentation shows that $T_1 = 1$ and $T_2 = 3$. For 3 disks, the solution given above proves that $T_3 \leq 7$. We can generalize the approach used for 3 disks to the following recursive algorithm for **n** disks.

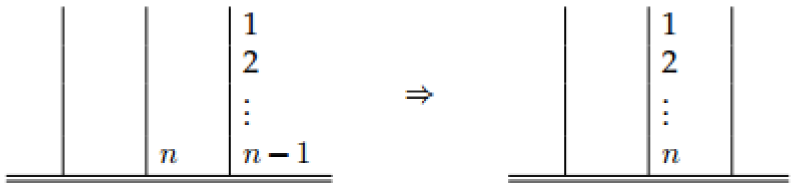
Step 1. Apply this strategy recursively to move the top n − 1 disks from the first post to the third post. This can be done in T_{n-1} steps



Step 2. Move the largest disk from the first post to the second post. This requires just 1 step.



Step 3. Recursively move the n−1 disks on the third post over to the second post. Again, T_{n-1} steps are required.



This algorithm shows that T_n, the number of steps required to move n disks to a different post, is at most $2T_{n-1} + 1$. We can use this fact to compute upper bounds on the number of steps required for various numbers of disks:

$$T_3 \leq 2.T_2 + 1$$

$$= 7$$

$$T_4 \leq 2.T_3 + 1$$

$$\leq 15$$

A Lower Bound for Towers of Hanoi:

For this to happen, the n − 1 smaller disks must all be stacked out of the way on the only remaining post. Arranging the n − 1 smaller disks this way requires at least T_{n-1} moves. After the largest disk is moved, at least another T_{n-1}moves are required to pile the n − 1 smaller disks on top. This argument shows that the number of steps required is at least $2T_{n-1} + 1$. Since we gave an algorithm using exactly that number of steps, we now have a recurrence for T_n, the number of moves required to complete the Tower of Hanoi problem with n disks:

$$T_1 = 1$$
$$T_n = 2T_{n-1} + 1 \qquad (for\ n \geq 2)$$

We can use this recurrence to conclude that $T_2 = 3, T_2 = 7, T_4 = 15, ...$

Check Your Progress 1

1. What is Recurrence relation?

2. Explain the concept of Tower of Hanoi.

6.4 HOMOGENEOUS LINEAR RECURRENCE RELATION:

6.4.1 CONCEPT:

A **linear recurrence relation of order** k with constant coefficient is a recurrence relation of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$, $c_k \neq 0$.

For example,

1. The relation $a_n = (-2)a_{n-1}$ is a linear homogeneous recurrence relation of order 1.

2. The recurrence relation $a_n = a_{n-1} + a_{n-2}$ is a linear recurrence relation of order 2.

3. The recurrence relation $a_n = 2na_{n-1}$ is **not** a linear recurrence relation with constant coefficients because the coefficient $2n$ is not constant. It is a linear homogeneous recurrence relation with **non-constant coefficients.**

4. The recurrence relation $a_n = a_{n-1} + 2$ is **not** a linear homogeneous recurrence relation because $a_n - a_{n-1} \neq 0$. It is an inhomogeneous recurrence relation.

5. The recurrence relation $a_n + 7a_{n-2} = 0$ is a **second order linear recurrence relation** with constant coefficients.

6. The recurrence relation $f_n = f_{n-1}^2 + f_{n-2}$ is **not** a linear homogeneous relation.

Definition: The equation $x^k = r_1 x^{k-1} + r_2 x^{k-2} + \ldots + r_k$ of degree k is called the **characteristic equation** of the linear homogeneous recurrence relation $a_n = r_1 a_{n-1} + r_2 a_{n-2} + \ldots + r_k a_{n-k}$
of order k.

6.4.2 THEOREM

Theorem:
If the characteristic equation $x^2 - r_1 x - r_2 = 0$ of the homogeneous recurrence relation $a_n = r_1 a_{n-1} + r_2 a_{n-2}$ has two distinct roots s_1 and s_2, then $a_n = us_1^n + vs_2^n$ where u and v depend on the initial conditions, is the explicit formula for the sequence.
(To say "u and v depend on the initial conditions" means that u and v are the solutions of the system of simultaneous equation $a_1 = us_1 + vs_2$ and $a_2 = us_1^2 + vs_2^2$)

Proof. Since s_1 and s_2 are roots of the characteristic equation $x^2 - r_1 x - r_2 = 0$, we have

$$s_1^2 - r_1 s_1 - r_2 = 0 \qquad (1)$$

$$s_2^2 - r_1 s_2 - r_2 = 0 \qquad (2)$$

Let $a_n = us_1^n + vs_2^n \quad for\ n \geq 1 \qquad (3)$

It is sufficient to show that (3) defines the same sequence as $a_n = r_1 a_{n-1} + r_2\, a_{n-2}$. We have

$$a_1 = us_1 + vs_2$$

$$a_2 = us_1^2 + vs_2^2$$

and the initial conditions are satisfied. Further,

$$\begin{aligned}
a_n &= us_1^n + vs_2^n \\
&= us_1^{n-2}.s_1^2 + vs_2^{n-2}.s_2^2 \\
&= us_1^{n-2}(r_1 s_1 + r_2) + vs_2^{n-2}(r_1 s_1 + r_2)[using\ (1) and\ (2)] \\
&= r_1 us_1^{n-1} + r_2 us_1^{n-2} + r_1 vs_2^{n-1} + r_2 vs_2^{n-2} \\
&= r_1(us_1^{n-1} + vs_2^{n-1}) + r_2(us_1^{n-2} + vs_2^{n-2}) \\
&= r_1 a_{n-1} + r_2 a_{n-2} \qquad [using\ expression\ from\ (3)]
\end{aligned}$$

Hence (3) defines the same sequence as $a_n = r_1 a_{n-1} + r_2 a_{n-2}$.

Hence $a_n = us_1^n + vs_2^n$ is the solution to the given linear homogeneous relation.

Theorem:

If the characteristic equation $x^2 - r_1 x - r_2 = 0$ of the linear homogeneous recurrence relation $a_n = r_1\, a_{n-1} + r_2\, a_{n-2}$ has a single root s, then the **explicit formula** (solution) for the recurrence relation is $a_n = us^n + vns^n$, where u and v depend on the initial conditions.

Proof. Since s is the root of the characteristic equation, we have

$$s^2 - r_1 s - r_2 = 0 \qquad (1)$$

Let

$$a_n = u\,s^n + v\,n\,s^n,\ n \geq 1. \qquad (2)$$

It suffices to show that (2) defines the same sequence as $a_n = r_1 a_{n-1} + r_2 a_{n-2}$. We have

$a_1 = u\,s + v\,s$,

$a_2 = u\,s^2 + 2v\,s^2$

and the initial conditions are satisfied. Also

Notes

an = u sn + v n sn

= u sn − 2 · s2 + v n sn − 2 · s2

= u sn − 2 (r1s + r2) + vnsn − 2 (r1s + r2) (using (1))

= r1 usn − 1 + r2 usn − 2 + r1 vnsn − 1 + r2 vnsn – 2

= r1 (usn − 1 + vnsn − 1) + r2(usn − 2 + vnsn − 2)

= r1an − 1 + r2an − 2 (using the expression for an − 1 and an − 2 from (2)).

Thus (2) defines the same sequence as $a_n = r_1a_{n-1} + r_2a_{n-2}$ and so is the explicit formula for the recurrence relation.

Example: Find an explicit formula for the sequence defined by the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$, n ≥ 2,with the initial conditions$a_0 = 1$ and $a_1 = 8$.

Solution: The recurrence relation $a_n = a_{n-1} + 2a_{n-2}$is a linear homogeneous relation of order 2.

Its characteristic equation is$x^2 - x - 2 = 0$which yields

$$x = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} = 2, -1$$

Hence,

$a_n = u(2)^n + v(-1)^n$ (1)

and we have

$a_0 = u + v = 1$ (given),

$a_1 = 2u - v = 8$ (given).

Solving for u and v, we have

$u = 3, v = -2$.

Hence, $a_n = 3(2)^n - 2(-1)^n$, $n \geq 0$is the explicit formula for the sequence.

Example: Solve the recurrence relation $d_n = 2d_{n-1} - d_{n-2}$ with initial conditions $d_1 = 1.5$ and $d_2 = 3$.

Solution:

The relation $d_n = 2d_{n-1} - d_{n-2}$ is a linear homogeneous recurrence relation of order 2. The characteristic equation (or associated equation) for this recurrence relation is $x_2 - 2x + 1 = 0$

which yields

$$x = \frac{2 \pm \sqrt{4-4}}{2} = 1{,}1$$

Thus, the characteristic equation has a multiple root 1.

Hence, $d_n = u \cdot 1^n + nv \cdot 1^n = (u + nv) \cdot 1^n$ and so

$d_1 = u + v = 1.5$ (given),

$d_2 = u + 2v = 3$ (given).

Solving for u and v, we get $u = 0$, $v = 1.5$ and so $d_n = 1.5n$ is the explicit formula (homogeneous solution) for the given recurrence relation.

6.4.3. Particular Solution Of Difference Equation

Definition: The (total) solution of a linear difference equation (linear recurrence relation)

$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \ldots + r_k a_{n-k} = f(n)$,where $f(n)$ is constant or a function of n with constant coefficients is the sum of two parts, the homogeneous solution satisfying the difference equation when the right-hand side of the equation is set to be 0, and the particular solution, which satisfies the difference equation with $f(n)$ on the right-hand side.

The particular solution is obtained by the method of inspection because we do not have a general procedure to find particular solution of a given difference equation. A general form of particular integral is set up according to the form of $f(n)$. We will consider the following cases

Case I: If $f(n)$ is a polynomial inn of degree m then we take $P_1 n^m + P_2 n^{m-1} + \ldots + P_{m+1}$as the particular solution of the difference equation. Putting this solution in the given difference equation, the values of $P_1, P_2 \ldots, P_{m+1}$ are determined.

Example: Find the particular solution of the difference equation

$a_n - a_{n-1} - 2a_{n-2} = 2n^2$.

Also write down the total solution.

Solution: Suppose that the particular solution is of the form

$$P_1 n^2 + P_2 n + P_3, \qquad (1)$$

where P_1, P_2 and P_3 are constants to be determined. Substituting (1) in the given difference equation, we obtain

$(P_1 n^2 + P_2 n + P_3) - [P_1 (n-1)^2 + P_2 (n-1) + P_3] - 2[P_1 (n-2)^2 + P_2 (n-2) + P_3] = 2n^2$

or

$P_1 n^2 + P_2 n + P_3 - [P_1 (n^2 + 1 - 2n) + P_2 (n-1) + P_3] - 2[P_1 (n^2 + 4 - 4n) + P_2 (n-2) + P_3] = 2 n^2$

or

$-2P_1 n^2 + n (10 P_1 - 2 P_2) + (-9 P_1 + 5 P_2 - 2 P_3) = 2 n^2$.

Comparing coefficients of the powers of n, we have

$-2 P_1 = 2$,

$10 P_1 - 2 P_2 = 0$,

$9 P_1 - 5 P_2 + 2 P_3 = 0$,

which yield

$P_1 = -1, P_2 = -5, P_3 = -8$.

Therefore, the particular solution is

$-n^2 - 5n - 8$.

The homogeneous solution of this recurrence relation is

$3(2)^n - 2(-1)^n$.

Hence the total solution is $3.2^n - 2(-1)^n - n^2 - 5n - 8$.

Example: Find the particular solution of the difference equation

$a_n + 5a_{n-1} + 6a_{n-2} = 3n^2 - 2n + 1$.

Solution: Here the right-hand side is a function of n and it is a polynomial of degree 2. So suppose that particular solution is of the form

$P_1 n^2 + P_2 n + P_3$, (1)

where P_1, P_2 and P_3 are to be determined. Substituting (1) in the given difference equation, we get

$P_1 n^2 + P_2 n + P_3 + 5[P_1 (n-1)^2 + P_2 (n-1) + P_3] + 6[P_1 (n-2)^2 + P_2 (n - 2) + P_3] = 3n^2 - 2n + 1$,which yields

$12P_1 n^2 - n(34P_1 - 12P_2) + (29P_1 - 17P_2 + 12P_3) = 3n^2 - 2n + 1$.

Comparing the coefficients of the powers of n we have

$12P_1 = 3$,

$34P_1 - 12P_2 = 2$,

$29P_1 - 17P_2 + 12P_3 = 1$

and so

$$P_1 = \frac{1}{4}, P_2 = \frac{13}{24}, and\ P_3 = \frac{71}{288}$$

Hence, the particular solution is

$$\frac{1}{4}n^2 + \frac{13}{24}n + \frac{71}{288}$$

Case II: If $f(n)$ is a constant, then the particular solution of the difference equation will also be a constant P, **provided that 1 is not a characteristic root of the difference equation**.

Example: Find the particular solution of the difference equation $a_n - 4\,a_{n-1} + 5a_{n-2} = 2$. Hence find the total solution of this recurrence relation.

Solution: Here $f(n) = 2$ (constant) and 1 is not characteristic root. So the particular solution will also be a constant P. Putting in the given recurrence relation of order 2, we have

$P - 4P + 5P = 2,$

$\Rightarrow 2P = 2,$

$\Rightarrow P = 1.$

Also, the characteristic equation of the given difference equation is

$x^2 - 4x + 5 = 0$, whose roots are

$$x = \frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i$$

Thus the homogeneous solution is

$u \cdot (2+i)^n + v \cdot (2-i)^n,\ n \geq 0.$

Hence the total solution of the given difference equation is

$a_n = u \cdot (2+i)^n + v \cdot (2-i)^n + 1.$

Case III: If $f(n)$ is of the form α^n, the corresponding particular solution is of the form $P\,\alpha^n$ provided that α is not a characteristic root of the difference equation of order n.

Example: Find the particular solution of the difference equation $a_n + 5\,a_{n-1} + 4a_{n-2} = 56 \cdot 3^n$. Hence, find the total solution of this difference equation?

Solution: The characteristic equation of the difference equation is

$x^2 + 5x + 4 = 0$

whose roots are -4 and -1. Thus the homogeneous solution is

$u\,(-4)^n + v\,(-1).$

Since $f(n) = 56 \cdot 3^n$ and 3 is not a characteristic root, the particular solution is of the form $P\,3^n$. Substituting this in the difference equation, we get

$$P.3^n + 5P(3)^{n-1} + 4P(3)^{n-2} = 56.3^n$$

$$\Longrightarrow P(3^n + 5.3^{n-1} + 4.3^{n-1} = 56.3^n)$$

$$\Longrightarrow P(3^n + \frac{5}{3}.3^n + \frac{4}{9}.3^n = 56.3^n)$$

$$\Rightarrow P\left(1 + \frac{5}{3} + \frac{4}{9}\right) = 56$$

$$\Rightarrow P\left(\frac{9 + 15 + 4}{9}\right) = 56$$

$$\Longrightarrow P = 18$$

Hence the particular solution is $18 \cdot 3^n$.

Case IV: If α is not a characteristic root of the difference equation and $f(n)$ is of the form

$(c_1 n^m + c_2 n^{m-1} + \ldots + c_{n+1})\, \alpha^n,$

then the particular solution is of the form

$(P_1 n^m + P_2 n^{m-1} + \ldots + P_{n+1})\, \alpha^n.$

Example: Find the total solution of the difference equation

$a_n - a_{n+1} - 2a_{n-2} = 3n \cdot 4^n.$

Solution: The characteristic equation of the given difference equation is

$x^2 - x - 2 = 0$

whose roots are 2 and -1.

Hence its homogeneous solution is

$u \cdot 2^n + v \cdot (-1)^n.$

Further, $f(n) = 3n \cdot 4^n$ and 4 is not a characteristic root of the difference equation. Hence particular solution is of the form

$(n\mathrm{P}_1 + P_2)\, 4^n.$ (1)

Substituting (1) in the given difference equation, we get

$$(nP_1 + P_2)4^n - [(n-1)P_1 + P_2]4^{n-1} - 2[(n-2)P_1 + P_2]4^{n-2} = 3n.4^n$$

$$\Longrightarrow (nP_1 + P_2)4^n - \frac{1}{4}[(n-1)P_1 + P_2]4^n - \frac{2}{16}[(n-2)P_1 + P_2]4^n = 3n.4^n$$

$$\Longrightarrow n.4^n\left(P_1 - \frac{1}{4}P_1 - \frac{1}{8}P_1\right) + 4^n\left(\frac{P_1}{2} + P_2 - \frac{P_2}{4} - \frac{P_2}{8}\right) = 3n.4^n$$

$$\Longrightarrow n.4^n\left(\frac{5P_1}{8}\right) + 4^n\left(\frac{P_1}{2} + \frac{5P_2}{8}\right) = 3n.4^n$$

Comparing coefficients of both sides, we have

$$P_1 = \frac{24}{5} and \ \frac{1}{2}P_1 + \frac{5}{8}P_2 = 0 \ or \ P_2 = -\frac{96}{25}$$

Hence the particular solution is

$$\left(\frac{24}{5}n - \frac{96}{25}\right)4^n$$

Therefore the total solution is

$$an \quad = u.2^n + v(-1)^n + \left(\frac{24}{5}n - \frac{96}{25}\right)4^n$$

Case V: If α is a characteristic root of multiplicity $m-1$ and $f(n)$ is of the form

$(c_1 n^p + c_2 n^{p-1} + \ldots + c_{p+1})\, \alpha^n$, the corresponding particular solution of the recurrence relation will be of the form

$n^{m-1}(P_1 n^p + P_2 n^{p-1} + \ldots + P_{p+1})\, \alpha^n$.

Example: Find the particular solution of the difference equation $a_n - 4a_{n-1} = 6 \cdot 4^n$.

Solution: The characteristic equation of the given difference equation is

$x - 4 = 0$

and so 4 is a root of multiplicity 1. Therefore, the particular solution is of the form

$n\,P \cdot 4^n$. (1)

Substituting (1) in the given difference equation, we get

$$nP \cdot 4^n - 4(n-1)P \cdot 4^{n-1} = 6 \cdot 4^n$$

$$\Rightarrow \ nP4^n - (n-1)P \cdot 4^n = 6 \cdot 4^n$$

$$\Rightarrow \ nP - nP + P = 6$$

$$\Rightarrow \ P = 6.$$

Hence the **Particular solution is** $6n \cdot 4^n$, whereas the **total solution** of the given difference equation is $4^n\,(u + 6n)$.

6.5 LINEAR NON-HOMOGENEOUS RECURRENCE RELATION:

A linear non-homogenous recurrence relation with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + f(n)$$

where $c_1, c_2, \ldots, c_k$are real numbers, and $f(n)$ is a function depending only on $\boldsymbol{n}$.

The recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$, is called the **associated homogeneous recurrence relation**. This recurrence includes k initial conditions. $a_0 = C_{0,} a_1 = C_{1,} \ldots a_k = C_{k,}$

Example: The following recurrence relations are linear nonhomogeneous recurrence relations

1. $a_n = a_{n-1} + 2n$
2. $a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$
3. $a_n = a_{n-1} + a_{n-4} + n!$
4. $a_n = a_{n-6} + n2^n$

THEOREM:

Let $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + f(n)$ be a linear non-homogeneous recurrence. Assume the sequence b_n satisfies the recurrence. Another sequence a_n satisfies the nonhomogeneous recurrence if and only if $h_n = a_n - b_n$ is also a sequence that satisfies the associated homogeneous recurrence is also.

Proof:

Part1: If h_n satisfies the associated homogeneous recurrence then a_n satisfies the non-homogeneous recurrence.

$$b_n = c_1 b_{n-1} + c_2 b_{n-2} + \cdots + c_k b_{n-k} + f(n)$$

$$h_n = c_1 h_{n-1} + c_2 h_{n-2} + \cdots + c_k h_{n-k}$$

Now

$$b_n + h_n = c_1(b_{n-1} + h_{n-1}) + c_2(b_{n-2} + h_{n-2}) + \cdots + c_k(b_{n-k} + h_{n-k}) + f(n)$$

Since,

$$a_n = b_n + h_n$$
$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + f(n)$$

So, a_n is a solution of the non-homogeneous recurrence.

Part2:If a_n satisfies the non-homogeneous recurrence then $h_n$is satisfies the associated homogeneous recurrence.

$$b_n = c_1 b_{n-1} + c_2 b_{n-2} + \cdots + c_k b_{n-k} + f(n)$$
$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + f(n)$$
$$a_n - b_n = c_1(a_{n-1} - b_{n-1}) + c_2(a_{n-2} - b_{n-2}) + \cdots + c_k(a_{n-k} - b_{n-k})$$
$$h_n = a_n - b_n$$

Since,

$$h_n = c_1 h_{n-1} + c_2 h_{n-2} + \cdots + c_k h_{n-k}$$

So, $h_n$is a solution of the associated homogeneous recurrence.

Proposition:

Let $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + f(n)$be a linear nonhomogeneous recurrence.

Assume the sequence $b_n$satisfies the recurrence.

Another sequence an satisfies the non-homogeneous recurrence ifand only if$h_n = a_n - b_n$is also a sequence that satisfies theassociated homogeneous recurrence.

Proof: We already know how to find h_n.

For many common$f(n)$, a solution b_n to the non-homogeneousrecurrence is similar to f(n).

Then you should find solution$a_n = b_n + h_n$to the nonhomogeneousrecurrence that satisfies both recurrence andinitial conditions.

Example: What is the solution of the recurrence relation$a_n = a_{n-1} + a_{n-2} + 3n + 1\ for\ n \geq 2$ with $a_0 = 2$ and $a_1 = 3$?

Solution: Since it is linear non-homogeneous recurrence, b_n is similar to $f(n)$

Guess: $b_n = cn + d$

$b_n = b_{n-1} + b_{n-2} + 3n + 1$

$cn + d = c(n-1) + d + c(n-2) + d + 3n + 1$

$cn + d = cn - c + d + cn - 2c + d + 3n + 1$

$0 = (3 + c)n + (d - 3c + 1)$

$c = -3, d = -10$

So $b_n = -3n - 10$

(bn only satisfies the recurrence, it does not satisfy the initial conditions.)

Check Your Progress 2

1. Define the terms:

   a.Associated homogeneous recurrence relation

   characteristic equation

   ___

   ___

   ___

2. What is total solution?

   ___

   ___

   ___

6.6 SEQUENCES AND RECURRENCE RELATION

Notes

We have encounter various types of 'numerical reasoning' puzzles where we have to fill in a missing number among a chain of numbers, such as 1, 4, 7, 10 …

Obviously, our answer would be 13, as in this sequence each term exceeds the previous one by 3 which is an example of a **recurrence relation**, in which each term has a certain relation with the previous term(s), and such a chain of numbers is called a **sequence**.

Also, we can replace 13 with any other number, the answer is still valid because we can always describe some kind of relations between the terms. We can a generalized formula by filling in the blank with any number, which gives the sequence when one enters 1, 2, 3, 4, … into the formula.

Consider the formula $3n - 2$, we can enter $n = 1, 2, 3, 4, 5 \ldots$ in the given formula and we get the sequence as 1, 4, 7, 10, 13 … . This popular method is known as Lagrange's interpolation, wherein we can actually find formulas in the form of polynomials which give sequences like 1, 4, 7, 10, 14 or with the fifth term replaced by any other number. Such a formula is said to give a **general term** for the sequence.

Let us consider one example to clarify our discussion and take it further.

Example: A piece of paper is 1 unit thick. By folding into half, the thickness becomes 2 units. Folding into half again, its thickness becomes 4 units, and so on.

(a) What is the thickness of the paper after it is folded 12 times?

(b) What is the thickness of the paper after it is folded 2004 times?

Solution:

i. When a piece of paper is folded 3 times, the thickness becomes 8 units. Similarly, when we fold it for 4, 5, 6 times, the thickness becomes 16, 32 and 64 units respectively. So, each time the paper is folded, its thickness doubles, that means we can simply multiply the thickness by 2 each time. Continuing in the same way, when it is folded 12 times, the thickness willbecome 4096 units.

ii. Here the case is different. We don't want to opt for the multiplication more than 2000 times and give the answer in a long chain of digits. As we had observe that the thicknesses 2, 4, 8, 16, etc. are actually powers of 2. Indeed, it is not difficult to see that after *n* folds, the thickness would be 2^nunits. Hence the answer is 2^{2004}

In the above example, if we let $\boldsymbol{x_n}$be the thickness of the paper when it is folded *n* times, then the two parts would be asking for the values of $\boldsymbol{x_{12}}$and $\boldsymbol{x_{2004}}$respectively. The value of $\boldsymbol{x_{12}}$may be calculated directly, but that of $\boldsymbol{x_{2004}}$is quite difficult. To compute $\boldsymbol{x_{12}}$we actually list out the values of x_1, x_2, x_3 … In this way we obtain a **sequence** of numbers.

Also, each time the paper is folded, its thickness doubles this implies $x_{n+1} = 2x_n$. This formula, which relates a term in the sequence with previous term(s), is known as a **recurrence relation** for the sequence. The fact that the initial thickness is 1 unit may be expressed as $x_0 = 1$. This is known as an **initial condition**. With the initial condition and the recurrence relation, we are able to compute $\boldsymbol{x_n}$ for small $\boldsymbol{n}$.
For large value of *n*, the computation will be clumsy and we want to write down the value of $a_n$neatly. So we can write $a_n = 2^n$, and such a formula is known as a **general term** of the sequence, which expresses the *n*-th term in terms of *n* (and independent of the previous terms of the sequence, unlike in the recurrence relation).

Example: Observe the following pattern.

$$1 = 1^3$$

$$3 + 5 = 2^3$$

$$7 + 9 + 11 = 3^3$$

$$13 + 15 + 17 + 19 = 4^3$$

$$\boxed{\qquad\qquad}$$

$$= 100^3$$

(a) How many ‘+’ signs should there be in the box?

(b) What are the smallest and largest integers in the box?

Solution:

(a) The number of ‘+’ signs in the first five rows are 0, 1, 2, 3, 4 respectively. Hence, if a_n denotes the number of ‘+’ signs in the *n*-th row, then we have $a_n = n - 1$, so that the number of ‘+’ signs in the 100th row is $a_{100} = 100 - 1 = 99$.

(b) Let b_n and c_n denote respectively the smallest and largest integers on the left hand side of the *n*-th row. Then we have to find b_{100} and c_{100} and it would be very easy if we can find the general term for b_n and c_n.

Note that for *n* = 1, 2, 3, 4, 5, b_n is equal to 1, 3, 7, 13, 21 respectively. To find an expression for b_n in terms of *n*, if we subtract 1 from each term, the sequence will become 0, 2, 6, 12, 20 … This is more evident, for they are 0×1, 1×2 , 2×3 , 3×4 and 4×5 respectively.

In this way, we see that $b_n = n(n-1) + 1$, so the smallest integer in the box is

$b_{100} = 100(100-1) + 1 = 9901.$

Now to find the general term for c_n is easy, for the smallest integer on the left hand side of the

$(n+1)^{st}$ row is b_{n+1}, so $c_n = b_{n+1} - 2 = n(n+1) + 1 - 2 = n^2 + n - 1.$

In particular, the largest integer in the box is

$$c_n = 100^2 + 100 - 1 = 10099$$

6.6.1 Finding The General Term Of A Sequence From The Recurrence Relation

A. INITIAL CONDITION:

This method is best illustrated with the help of following examples as shown below:

Example: A staircase consists of *n* steps. A boy walks from the bottom to the top, each time climbing 1 or 2 steps.

(a) If *n* = 10, what is the number of ways in which he can climb up the stairs?

(b) If *n* = 2004, what is the number of ways in which he can climb up the stairs?

Solution:

Let x_n be the number of ways to climb up a stairs of *n* steps with each time climbing 1 or 2 steps.

So we have to compute x_{10} and x_{2004}. To climb *n* steps, we may first climb 1 or 2 steps. In the former case, we are to climb *n* −1 more steps, and this can be done in x_{n-1}ways. In the latter case, we are to climb *n* −2 more steps, and this can be done in x_{n-2}ways. Therefore, we obtain the recurrence relation $x_n = x_{n-1} + x_{n-2}$

Since the recurrence relation for $x_n$depends on two previous terms, we need two initial conditions i.e. $x_1 = 1$and $x_2 = 2$.

By direct computation, we can obtain $x_{10} = 89$, thus solving part (a).

For part (b) it's difficult to have direct computation and to find the general term.

So we will solve another example as follows that will help with the (b) part solution as well.

Example: For positive integer *n*, let

$$f(n) = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)$$

where α and β $(\alpha > \beta)$are roots of the equation.

$$x^2 - x - 1 = 0$$

(a) Find $\alpha + \beta, \alpha\beta, f(1)$ and $f(2)$

(b) Show that $f(n+2) = f(n) + f(n+1)$

(c) Show that $f(n)$ is an integer for al positive integer n.

Solution:

(a) The value of $\alpha + \beta$ is the sum of roots of the equation $x^2 - x - 1 = 0$, which is 1.

Similarly, $\alpha\beta$ is the product of the roots, which is -1.

We also have

$$f(1) = \frac{1}{\sqrt{5}}(\alpha - \beta) = \frac{1}{\sqrt{5}} \cdot \sqrt{(\alpha+\beta)^2 - 4\alpha\beta} = \frac{1}{\sqrt{5}}[(1)^2 - 4(-1)]$$

$$= 1$$

$$f(2) = \frac{1}{\sqrt{5}}(\alpha^2 - \beta^2) = \frac{1}{\sqrt{5}}(\alpha - \beta)(\alpha + \beta) = f(1).(\alpha + \beta)$$

$$= 1 \times 1 = 1$$

Since α and $\beta$are roots of the equation $x^2 - x - 1 = 0$, we have $\alpha^2 - \alpha - 1 = 0$ i.e. $\alpha^2 = \alpha + 1$.

Similarly, $\beta^2 = \beta + 1$. Consequently,

$$\sqrt{5}f(n+2) = \alpha^{n+2} - \beta^{n+2}$$
$$= \alpha^n.\alpha^2 - \beta^n.\beta^2$$
$$= \alpha^n(\alpha+1) - \beta^n(\beta+1)$$
$$= (\alpha^{n+1} - \beta^{n+1}) + (\alpha^n - \beta^n)$$
$$= \sqrt{5}.f(n+1) + \sqrt{5}.f(n)$$

Dividing both sides by $\sqrt{5}$, we get the desired result as follows

$$f(n+2) = f(n) + f(n+1)$$

(c) Since *f* (1) and *f* (2) are integers, and that $f(n+2) = f(n) + f(n+1)$ so *f* (*n*) is an integer for all positive integers *n*.

Incidentally, the *f* (*n*) here is precisely the x_n in the above example. Therefore, the answer to part (b) in above example would be

$$x_{2004} = f(2004) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{2004} - \left(\frac{1-\sqrt{5}}{2}\right)^{2004}\right]$$

where the values of α and β are obtained by solving the equation $x^2 - x - 1 = 0$.

B.THE METHOD OF FINITE DIFFERENCES:

When polynomial is given and we are aware of first few terms, then it is easy to find the general term. For instance, let us consider the following ‘numerical reasoning’ puzzle:

7, 8, 12, 19, 29,________,….

After observing the above sequence we can easily make out that the difference between successive terms are 1, 4, 7 and 10. Obviously, the next difference would be 13, so the missing term is 42.

The next difference is 'naturally' 13, because then the 'second level differences' would be constantly equal to 3.

7 8 12 19 29

1 4 7 10

3 3 3

After two levels of taking differences, we have reached a 'finite difference', which is constantly 3. If we regard this as a continuous function, say $f(1) = 7, f(2) = 8, f(3) = 12$ and so on, and taking differences like this is same like doing differentiation. If the function becomes constant after taking differences twice, it means that the second derivative of the function is constant, and hence the function is a polynomial of degree 2. In this way, the general term of the above sequences to be of degree 2.

Let us consider the following expression

$$a_n = A_{n^2} + B_n + C$$

For some constants A, B and C, if we put $a_1 = 7, a_2 = 8\ and\ a_3 = 12$, we obtain

$$7 = A + B + C$$

$$8 = A + 2B + C$$

$$12 = 9A + 3B + C$$

We have three linear equations in three unknowns, we can solve these for the values of A, B and C. One of the easiest way to solve the above system of equation is to take successive differences of the equations. For example, if we subtract the first equation from the second and the second from the third, we get the following equation

$$1 = 3A + B$$

$$4 = 5A + B$$

If we again take the difference, we have 3 = 2A, so A= 3/2. Backward substitution yields B = – 7/2 and C = 9. The general term for the sequence is given as

$$a_n = \frac{3}{2}n^2 - \frac{7}{2}n + 9$$

C. Backtracking Method:

CONCEPT: Consider the sequence 1, 4, 9, 16, 25, 36, 49 … which is a sequence of the squares of all positive integers. We can define this sequence by the formula $a_n = n^2$, $1 \le n < \infty$.
So we have used only positive number to describe the terms of the sequence. Such type of formula is called **Explicit formula.**
Also, the explicit formula $a_n = (-4)^n$, $1 \le n < \infty$ describes the infinite sequence −4, 16, −64, 256, …

Example: Find the explicit formula for the finite sequence 87, 82, 77, 72, 67. Can this sequence be described by a recursive relation?
Solution: The explicit formula for the given finite sequence is $a_n = 92 - 5n$, $n = 1, 2$ ….
Also, it can be described by the recursive formula $a_1 = 87$, $a_n = a_{n-1} - 5$, $2 \le n \le 5$.

To study general properties of sequences, the recurrence relation with initial conditions are solved to get explicit formula. Such an explicit formula is called a **solution** of the given recurrence relation.

CONCEPT: A technique for finding an explicit formula for the sequence defined by a recurrence relation is **called backtracking.** In this technique, the values of a_n are back tracked, substituting the values of a_{n-1}, a_{n-2} and so on, till a pattern is clear.

Example: Find an explicit formula for the recurrence relation

$a_0 = 1$, $a_n = a_{n-1} + 2$.

Solution: The recurrence relation

$a_0 = 1$, $a_n = a_{n-1} + 2$ defines the sequence 1, 3, 5, 7, ….

We backtrack the value of a_n by the substituting the definition of a_{n-1}, a_{n-2} and so on untill there is a pattern. We have

$$
\begin{aligned}
a_n &= a_{n-1} + 2 \\
&= a_{n-2} + 2 + 2 = a_{n-2} + 2 \cdot 2 \\
&= a_{n-3} + 2 + 2 + 2 = a_{n-3} + 2 \cdot 3 \\
&= a_{n-4} + 2 + 2 + 2 + 2 = a_{n-4} + 2 \cdot 4 \text{ and so on.}
\end{aligned}
$$

Thus, backtracking yields

$a_n = a_{n-k} + 2k.$

If we set $k = n$, then

$a_n = a_{n-n} + 2\,n = a_0 + 2\,n = 1 + 2\,n$, which is the required **explicit formula.**

Example: Backtrack to find explicit formula for the sequence defined by the recurrence relation

$a_1 = 1,\ a_n = 3\,a_{n-1} + 1,\ n \geq 2.$

Solution: The recurrence relation defines the sequence 1, 4, 13, 40, ….

Backtracking yields

$$
\begin{aligned}
a_n &= 3\,a_{n-1} + 1 \\
&= 3(3\,a_{n-2} + 1) + 1 = 3^2 \cdot a_{n-2} + 3^1 + 1 \\
&= 3\{3(3\,a_{n-3} + 1) + 1\} + 1 = 3^3\,a_{n-3} + 3^2 + 3^1 + 1 \\
&= 3[3\{3(3\,a_{n-4} + 1) + 1\} + 1] + 1 \\
&= 3^4\,a_{n-4} + 3^3 + 3^2 + 3^1 + 1 \text{ and so on.}
\end{aligned}
$$

The backtracking will end at

$a_n = 3^k\,a_{n-k} + 3^{k-1} + 3^{k-2} + \ldots + 3^2 + 3^1 + 1.$

If we set $k = n - 1$, then we have

$$a_n = 3^{n-1}\, a_{n-(n-1)} + 3^{n-2} + \ldots + 3^3 + 3^2 + 3^1 + 1$$

$$= 3^{n-1}\, a_1 + 3^{n-2} + \ldots + 3^3 + 3^2 + 3^1 + 1$$

$$= 3^{n-1} + 3^{n-2} + \ldots + 3^3 + 3^2 + 3^1 + 1$$

$$= \frac{3^n - 1}{3 - 1} = \frac{3^n - 1}{2}$$

Hence $a_n = \frac{3^n - 1}{2}$ is the required explicit formula.

6.7 DIFFERENCE TABLE:

CONCEPT: Suppose a sequence is given as $S = \{S_0, S_1 \ldots S_n\}$, let $\Delta\, ^j S_i$ denote the i[th] difference at the j[th] level, defined by the recurrence relation as

$$\Delta^m S_i = \Delta^{m-1} S_{i+1} - \Delta^{m-1} S_i$$
$$\Delta\, ^0 S_i = S_i$$

A difference table represents the finite difference of a sequence S as a triangular matrix as shown below:

$$\begin{array}{cccccc}
x_0 & S_0 & & & & \\
 & & \Delta S_0 & & & \\
x_1 & S_1 & & \Delta^2 S_0 & & \\
 & & \Delta S_1 & & \Delta^3 S_0 & \\
x_2 & S_2 & & \Delta^2 S_1 & & \Delta^4 S_0 \\
 & & \Delta S_2 & & \Delta^3 S_1 & \\
x_3 & S_3 & & \Delta^2 S_2 & & \\
 & & \Delta S_3 & & & \\
x_4 & S_4 & & & &
\end{array}$$

The height of a difference table is the smallest integer k such that $\Delta\, ^0 S_i = 0$ for $0 \le i \le n - k$

Notes

Consider a k-th degree polynomial $P(n,k) = a_0 n^k + a_1 n^{k-1} + \cdots + a_k$.
With reference to Binomial theorem

$$(n+1)^k - n^k = \sum_{i=0}^{k-1} \binom{k}{i} n^i$$

So the first order differences of P(n, k) is given as follows

$$P(n+1,k) - P(n,k) = a_0 \sum_{i=0}^{k-1} \binom{k}{i} n^i + \sum_{i=1}^{k-1} a_i \left[(n+1)^{k-i} - n^{k-i}\right]$$

define a polynomial of degree k – 1 where the coefficient of the leading term $n^k - 1$ is $a_0(k-1)$. If the difference table is given to us, we can easily reconstruct the polynomial defining the given sequence using Newton forward difference formula.

$$P(n,k) = \sum_{i=0}^{k} \binom{n}{i} \Delta^{\,i} S_0$$

6.8 LINE IN A PLANE

We will refer the following figures to understand the concept:

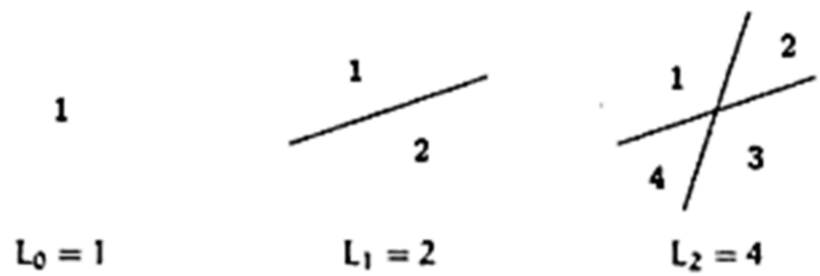


Fig 6.3: Line in Plane

The first plane has no lines so it has one region represented by L_0; the second plane has one line so it has two region represented by L_1 and third plane has a two line so it has four region represented by L_2

[**Note**: Each line extend infinitely in both the directions]

Now we can generalize it as $L_n = 2^n$ which implies that if we add a new line then the number of region get double. This generalization is false. We could achieve the doubling of the region if the nth line would split each old region in two; if each old region is convex.

[NOTE: A straight line can split a convex region into at most two new regions, which will also be convex. A region is convex if it includes all line segments between any two of its points.]

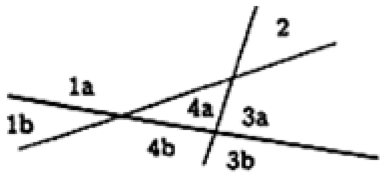


Fig 6.4: 3 Lines intersecting in a plane

But when we will add third line as shown in above diagram, the dark one we observe that it can split at most three of the old regions, no matter how the first two lines are placed in a plane.

Thus $L_3 = 4 + 3 = 7$. The nth line (for n > 0) increases the number of regions by k if and only if it splits k of the old regions, and it splits k old regions if and only if it hits the previous lines in k – 1 different places. Two lines can intersect in at most one point. Therefore the new line can intersect the n – 1 old lines in at most n – 1 different points, and we must have $k \leq n$.

We have established the upper bound

$L_n \leq L_{n-1} + n,$ for n > 0.

With the help of induction equality can be achieved in this formula.

Consider the situation where nth line is place in such a way that it's not parallel to any of the other line (this means it intersect them all) and such that it doesn't go through any of the existing intersection points (hence it intersects them all in different places). Therefore, the recurrence is given as

$L_0 = 1;$

$L_n = L_{n-1} + n$ for n > 0.

The known values of L_1, L_2 and L_3 satisfy the above relation. We cab write

$$L_n = L_{n-1} + n$$
$$= L_{n-2} + (n-1) + n$$
$$= L_{n-3} + (n-2) + (n-1) + n$$
$$.$$
$$.$$
$$.$$
$$= L_0 + 1 + 2 + \cdots + (n-2)(n-1) + n$$
$$= 1 + S_n$$

Where $S_n = 1 + 2 + 3 + \cdots + (n-1) + n$

In other words, L_n is one more than the sum S_n of the first n positive integers. We can have following table for S_n.

n	1	2	3	4	5	6	7	8	9	10
S_n	1	3	6	10	15	21	28	36	45	55

These values are also known as triangular numbers, because S_n is the number of bowling pins in an n-row triangular array. For instance four row array has $S_{10} = 10$ pins.

If we add S_n to its reversal, so that each of the n columns on the right sums to n + 1.

$S_n = 1 + 2 + 3 + \cdots + (n-1) + n$

$+S_n = n + (n-1) + (n-2) + \cdots + 2 + 1$

$2S_n = (n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1)$

$2S_n = n(n+1)$

$$S_n = \frac{n(n+1)}{2} \quad for\ n \geq 0$$

We have our final solution as

$$L_n = \frac{n(n+1)}{2} + 1\ for\ n \geq 0$$

This is the general equation for line in a plane.

CHECK YOUR PROGRESS 3

1. Explain the Finite method of finding general term.

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2. Enumerate the concept of back tracking.

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6.9 LET'S SUM UP

Recurrence relation breaks the bigger problem into smaller parts in running time so its best tool for computer algorithms. It has got application in digital signal processing as recurrence can help to model the feedback into the system. Linear recurrence relation is widely used in economics.

6.10 KEYWORDS

1. Recurrence sequence - The sequence or series generated by recurrence relation
2. Triangular numbers - any of the series of numbers (1, 3, 6, 10, 15, etc.) obtained by continued summation of the natural numbers 1, 2, 3, 4, 5, etc.
3. Polynomial - consisting of several terms
4. Explicit formula - is a **formula** we can use to find the nth term of a sequence

6.11 QUESTIONS FOR REVIEW

1. Derive recurrence relation for obtaining the amount A_n at the end of n years on the investment of Rs 10,000 at 5% interest compounded annually.
2. Developing a recurrence formula, find the number of bit strings of length 4 that do not contain the pattern 111.

3. Solve the recurrence relation

$$p_n = a - \frac{b}{k} . p_{n-1}$$

for the price in the economics model, where a, b, k are positive parameters and p_0 is the initial price.

4. Suppose that the population of a village is 100 at time $n = 0$ and 110 at time $n = 1$. The population increases from time $n - 1$ to time n is twice the increase from time $n - 2$ to time $n - 1$. Find a recurrence relation and the initial conditions for the population at time n and then find the explicit formula for it.

5. Find the particular solution of the difference equation $a_n + 5a_{n-1} + 6a_{n-2} = 3n^2 - 2n + 1$.

6.12 SUGGESTED READINGS

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6.13 ANSWER TO CHECK YOUR PROGRESS

1. Explain the concept with example – 6.1
2. Explain the complete concept – 6.2
3. State the concept (a) --6.4 & (b) – 6.3.1.2
4. State the concept 6.3.3
5. Explain the concept --- 6.5.1 - B
6. Explain the concept --- 6.5.1 - C

UNIT 7: RECURRENCE RELATION AND GENERATING FUNCTION

STRUCTURE

7.0 OBJECTIVE

- What is a Recurrence Relation?
- Generating function of sequence
- Generating function models
- The Fibonacci Relation and its Properties

- Other Recurrence Relation Model
    - Solving Recurrence Relation by Substitution and Generating Functions
    - The Method of Characteristics Roots
    - Distinct Roots and Multiple Roots

7.1 GENERATING FUNCTION OF SEQUENCE:

The concept of generating functions is a powerful tool for solving counting problems. Intuitively put, its general idea is as follows. In counting problems, we are often interested in counting the number of objects of 'size n', which we denote by a_n. By varying n, we get different values of a_n. In this way we get a sequence of real numbers

$$a_0, a_1, a_2, \ldots$$

Here our interest is in the sequence of real numbers ($a_0, a_1, a_2, \ldots, a_r, \ldots$), and such function whose domain is the set of nonnegative integers and whose range is the set of real numbers.

Expressions like $A = \{a_r\}_r^\infty$ is used to denote such sequences, where a_r is the number of ways to select **r** objects in some procedure.

Example 1: The sequence ${}_0A = \{2^r\}_{r=0}^\infty$ is the sequence (1,2,4,8,16, …, 2^r, …); the sequence $B = \{b_r\}_{r=0}^\infty$
Where

$$b_r = \begin{cases} 0 & if\ 0 \leq r \leq 4 \\ 2 & if\ 5 \leq r \leq 4 \\ 3 & if\ r = 10 \\ 4 & if\ 11 \leq r \end{cases}$$

Thus, B = (0, 0, 0, 0, 0, 2, 2, 2, 2, 2, 3, 4, 4 …)

$C = \{C_r\}_{r=0}^{\infty}$,where $C_r = r + 1$ for each value of r, is the sequence (1, 2, 3, 4, 5**…**)

and the sequence$D = \{d_r\}_{r=0}^{\infty}$where for each r $d_r = r^2$ is the sequence **(0, 1, 4, 9, 16, 25 …)**

- To the Sequence $A = \{a_r\}_{r=0}^{\infty}$, we will assign the symbol

$$A(X) = a_0 + a_1X + \cdots + a_nX^n = \sum_{n=0}^{\infty} a_rX^r$$

- The expression A (X) is called a **formal power series**, a_i is the **coefficient** of X^i , the term $a_i X^i$ is a term of degree I, and the term $a_0 X^0 = 0$ is called the **constant term.**

- The formal power series $A(X) = \sum_n^{\infty} a_rX^r$is called the generating function for the sequence$A = \{a_r\}_{r=0}^{\infty}$
- We will use the word 'formal' to distinguish between the abstract symbol
  $A(X) = \sum_n^{\infty} a_rX^r$ and the concept of Power series

Example 2: The generating functions of example 1

$$A(X) = \sum_{n=0}^{\infty} 2^rX^r$$

$$B(X) = 2X^5 + 2X^6 + 2X^7 + 2X^8 + 2X^9 + 3X^{10} + 4X^{11} + 4X^{12} + \dots$$

$$C(X) = \sum_{r=0}^{\infty}(r+1)X^r$$

$$D(X) = \sum_{r=0}^{\infty} r^2X^r$$

Let $A(X) = \sum_{r=0}^{\infty} a_r X^r$, $B(X) = \sum_{s=0}^{\infty} b_s X^s$ be two formal power series then we can define following concept as follows:

EQUALITY: A (X) = B (X) if and only if $a_n = b_n$ for each $n \geq 0$.

MULTIPLICATION BY A SCALAR NUMBER C: $CA(X) = \sum_{r=0}^{\infty} (Ca_r) X^r$

SUM: $A(X) + B(X) = \sum_{n=0}^{\infty} (a_n + b_n) X^n$

PRODUCT: $A(X)B(X) = \sum_{n=0}^{\infty} P_n X^n$

- $P_n X^n$ is the product of A(X) B (X) which is obtained by taking the sum of all possible products of one term from A(X) and one term from B(X) such that **n** = sum of the exponents
- Thus, this can be accomplished by considering a_0 which is a constant term of A (X) and multiply it with the coefficients b_n of X^n in B(X)

- Proceeding ahead, now coefficient $\mathbf{a_1}$ of X in A (X) and multiply it by the coefficient b_{n-1} of X^{n-1} in B(X) and so on.
- In product we are supposed to use the increasing powers of X in A(X) while decreasing powers of X in B(X)
- So we get

$P_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \ldots + a_n b_0 = \sum_{i=0}^{\infty} a_i b_{n-i}$

Thus,

$A(X)B(X) = a_0 b_0 + (a_0 b_1 + a_1 b_0) X + (a_0 b_2 + a_1 b_1 + a_2 b_0) X^2 + \ldots + (a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \ldots + a_n b_0) X^n + \ldots$

Example: If $S(X) = a_0 + a_2X^2 + a_4X^4 + a_8X^8$ and

$T(X) = b_0 + b_4X^4 + b_6X^6 + b_8X^8$ (we assume the coefficients for the missing power of X is zero)

- Then we can find the coefficient of X^r in S(X) T(X) by considering the powers { X^0, X^2, X^4, X^8 } from the first factor and the powers { X^0, X^4, X^6, X^8 } from the second factor such that there sum is **r**.
- For instance the coefficient of X^8 can be obtained by using X^0 in the first factor and X^8 in the second; X^2 in the first factor and X^6 in the second.
- Thus, the coefficient of X^8 in the product S(X) T(X) is such that
- $P_8 = a_0b_8 + a_2b_6 + a_4b_4 + a_8b_0$, because (0, 8), (2, 6), (4, 4), and (8, 0) are the only pairs of exponents of S(X) and T(X) whose sum is 8.
- Likewise the coefficient of X^6 in the product is $a_0 b_6 + a_2 b_4$, because there are only two pair of exponent of S(X) and T(X) , whose sum is 6.

Thus, if $a_0 = 2$, $a_2 = -5$, $a_4 = 7 + a_8 = 3$

$b_0 = 3$, $b_4 = -6$, $b_6 = 8$, $b_8 = 3$ then

$P_8 = (2)(3) + (-5)(8) + (7)(-6) + (3)(3) = -67$,

The case where the entire non-zero coefficients are 1 is of special interest.

We will have $P_8 = 4$

We can now conclude that the coefficient of X^8 in the product is just the number of pairs of exponents whose sum is 8 or in other words the number of integral solutions to the equation $e_1 + e_2 = 8$, where $\mathbf{e_1}$ and $\mathbf{e_2}$ represent the exponents of S(X) and T(X), respectively.

Remarks:

a. The coefficient of X^r in the product $(1 + X^2 + X^4 + X^8)(1 + X^4 + X^6 + X^8)$ is the number of integral solution to the equation $e_1 + e_2 = r$ subject to the constraints $e_1 = 0,2,4,8$ and $e_2 = 0, 4, 6, 8$.

b. The exponents of the factors in the product reflect the constraint in the equation.

c. We can compute the coefficient of X^r by algebra and then discover the number of integral solutions to the equation $e_1 + e_2 = r$ subject to the constraints ; or

d. We can compute all the solutions of the equation subject to the constraints and then discover the coefficient of X^r.

Check Your Progress 1

1. What do you understand by Generating Functions?

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2. Explain the concept of Power series and Equality

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7.2 GENERATING FUNCTION MODELS:

- ✓ Let us consider the product of generating function A(X) and B(X), where the exponents of A(X) reflect the constraints on $\mathbf{e_1}$and the exponents of B(X) reflect the constraints on $\mathbf{e_2}$.
- ✓ X^r is the coefficient of the product of generating function A(X) B(X)
- ✓ Assume $\mathbf{e_1}$ can only be 0, 1, 9 then let $A(X) = 1 + X + X^9$**.** If e_2 can only be even and $0 \le e_2 \le 8$,then $B(X) = 1 + X^2 + X^4 + X^6 + X^8$

- ✓ And if $\mathbf{e_1}$can be any non-negative integer value, then we let 1 + X + X^2 + …[A(X) has infinite number of terms] . Also, if $\mathbf{e_2}$ can only take on the integral values like that are multiples of 5, then we let B (X) = 1 + X^5 + X^{10} + … Hence, we can see endless possibilities.

Now we can extend the definition of product of formal power series for 3 factors as below:

$$A(X) = \sum_{i=0}^{\infty} a_i X^i$$

$$B(X) = \sum_{j=0}^{\infty} b_j X^j$$

$$C(X) = \sum_{k=0}^{\infty} c_k X^k$$

Then

$$A(X)B(X)C(X) = \sum_{r=0}^{\infty} P_r X^r$$

where

$$P_r = \sum_{i+j+k=r} a_i b_j c_k$$

Thus, we can find the term $P_r X^r$ by taking any one term $a_i X^i$ from A(X), any one term $b_j X^j$ from B(X) and any one term $c_k X^k$ from C(X) such that the sum of exponents i + j + k = r.
Assume each non zero coefficient of each formal power series $A_i(X)$ is 1. Then we have the coefficient of X^r in the product $A_1(X)\ A_2(X) \ldots A_n(X)$ indicates the number of integral solution to the equation $e_1 + e_2 + \ldots + e_n = r$ where constraints on each e_i is determined by the exponents of the i^{th} factor of $A_i(X)$.

Conversely if we are supposed to count the number of non-negative integral solutions to an equation $e_1 + e_2 + \dots + e_n = r$ with constraints on each e_i then we can build a generating function $A_1(X)\ A_2(X) \dots A_n(X)$ whose coefficient of X^r is the answer.

Example 1: Find a generating function for a_r = the number of non- negative integral solutions of $e_1 + e_2 + e_3 + e_4 + e_5 = r$ where $0 \le e_1 \le 3$, $0 \le e_2 \le 3$, $2 \le e_3 \le 6$, $2 \le e_4 \le 6$, e_5 is odd, and $1 \le e_5 \le 9$.

Let $A_1(X) = A_2(X) = 1+ X + X^2 + X^3$, $A_3(X) = A_4(X) = X^2 + X^3 + X^4 + X^5 + X^6$ and

$A_5(X) = X + X^3 + X^5 + X^7 + X^9$. Thus, the generating function we want is

$$A_1(X)\ A_2(X)\ A_3(X)\ A_4(X)\ A_5(X) = (1+ X + X^2 + X^3)^2 (X^2 + X^3 + X^4 + X^5 + X^6)^2 (X + X^3 + X^5 + X^7 + X^9).$$

Example 2: Find a generating function for a_r = the number of non- negative integral solutions of $e_1 + e_2 + \dots + e_n = r$ where $0 \le e_i \le 1$.

Let $A_i(X) = 1 + X$ for each $i = 1, 2, \dots, n$. Thus, the generating function we want is $A_1(X)\ A_2(X) \dots A_n(X) = (1 + X)^n$.

As per the binomial theorem the answer to above is C(n, r) – the coefficient of X^r term.

Example 3: Find the coefficient of X^{16} in $(1 + X^4 + X^8)^{10}$.

The only solution to $e_1 + e_2 + \dots + e_{10} = 16$ where $e_i = 0, 4, 8$ are those with four 4's, no 8's and six 0's or two 8's, no 4's and eight 0's; or two 4's, one 8, and seven 0's. Thus, the coefficient is

$$\binom{10}{4} + \binom{10}{2} + 8\binom{10}{2} = \binom{10}{4} + 9\binom{10}{2}$$

Example 4: Build a generating function for determining the number of ways of making change for a dollar bill in pennies, nickels, dimes, quarters, and half- dollar pieces. Which coefficient do we want?

We can find the coefficient of X^{100} in the product of

$$(1 + X + X^2 + \ldots + X^{100})$$
$$(1 + X^5 + X^{10} + \ldots + X^{100})$$
$$(1 + X^{10} + X^{20} + \ldots + X^{100})$$
$$(1 + X^{25} + X^{50} + X^{75} + X^{100})$$
$$(1 + X^{50} + X^{100}).$$

Example 5: Find a generating function for the sequence $A = \{a_r\}_{r=0}^{\infty}$ where

$$A = \begin{cases} 1 & \text{if } 0 \le r \le 2 \\ 3 & \text{if } 3 \le r \le 5 \\ 0 & \text{if } r \ge 6 \end{cases}$$

Solution: $1 + X + X^2 + 3X^3 + 3X^4 + 3X^5$

7.3 RECURRENCE RELATIONS:

CONCEPT: A recurrence relation is a formula that relates for any integer n ≥ 1, the n-th term of a sequence $A = \{a_r\}_{r=0}^{\infty}$ to one or more of the terms a_0, $a_1, \ldots a_{n-1}$.

Examples:

1. If s_n denotes the sum of the first n positive integers, then $s_n = n + s_{n-1}$ is a recurrence relations. Other examples are:
2. If d is a real number, then the n^{th} term of an arithmetic progression with common difference **d** satisfies the relation $a_n = a_{n-1} + d$.
3. If p_n denotes the n^{th} term of a geometric progression with common ratio **r,** $p_n = r\, p_{n-1}$ is a recurrence relation.

CONCEPT:

- Let n and m are non-negative integers. A recurrence relation of the form
- $c_0(n)\, a_n + c_1(n)\, a_{n-1} + \ldots + c_m(n)\, a_{n-m} = f(n)$ for $n \ge m$, where $c_0(n), c_1(n), \ldots, c_m(n)$ and $f(n)$ are functions of ***n*** is said to be ***Linear Recurrence Relation.***

- If c_0 (n) and c_m (n) are not identically zero, then it is said to be ***Linear Recurrence Relation of degree m.***

- If c_0 (n), c_1 (n), **…**, c_m (n) are constants, then it is known as ***Linear Recurrence Relation with constant coefficient.***

- If ***f*(n)** is identically zero then it is called as ***Homogenous***; Otherwise ***Inhomogenous.***

Example: $a_n - 9\, a_{n-1} + 26\, a_{n-2} - 24\, a_{n-3} = 5n$ ***Linear Recurrence Relation with constant coefficient.***

$p_n = r\, p_{n-1}$ ***Linear Recurrence Relation of degree 2***

$a_n - 4\, a_{n-1} + 2\, a_{n-2} = 0$ ***Homogenous***

7.3.1 Divide And Conquer Recurrence Relation:

A divide-and-conquer algorithm consists of three steps:

- dividing a problem into smaller sub-problems
- solving (recursively) each sub-problem
- then combining solutions to sub-problems to get solution to original problem.

We use recurrences to analyze the running time of such algorithms. Suppose T_n is the number of steps in the worst case needed to solve the problem of size *n*. Let us split a problem into $a \geq 1$ sub-problems, each of which is of the input size n/b where $b > 1$.

Observe, that the number of sub-problem ***a*** is not necessarily equal to ***b***. The total number of steps T_n is obtained by all steps needed to solve smaller sub-problems $T_{n/b}$ plus the number needed to combine solutions into a final one. The following equation is called

Notes

Divide-and-conquer recurrence relation

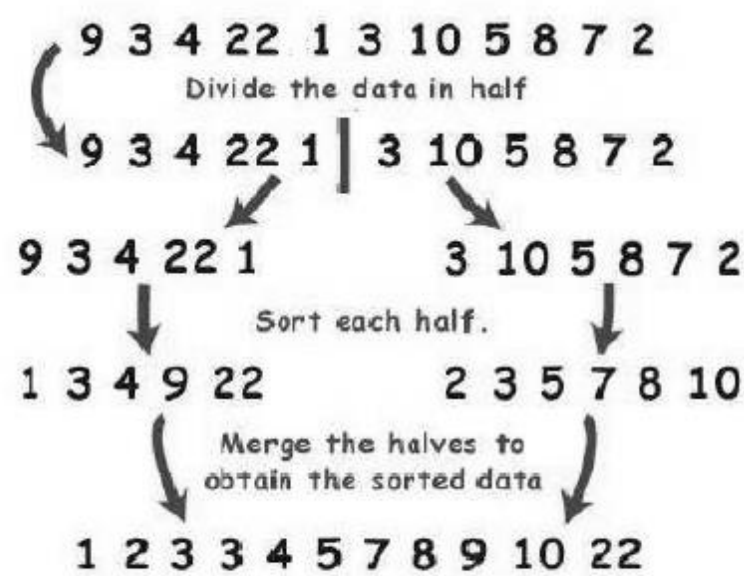
$$T_n = aT_{n/b} + f(n)$$

As an example, consider the mergesort:

-divide the input in half

-recursively sort the two halves

-combine the two sorted subsequences by merging them.



Let $\boldsymbol{T_n}$ be worst-case runtime on a sequence of $\boldsymbol{n}$ keys:

If $n = 1$, then $T_n = \Theta\ (1)$ constant time

If $n > 1$, then $T(n) = 2T\left(\frac{n}{2}\right) + \Theta(\mathrm{n})$

here $\Theta(\mathrm{n})$ is time to do the merge. Then

$$T(n) = 2T_{n/2} + \Theta(\mathrm{n})$$

Check your progress 2

1. What is Linear Recurrence Relation?

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2. Explain Divide and Conquer Relation

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7.4 SOLUTION OF RECURRENCE RELATION:

- ✓ To solve an equation like $x^2 – 7x + 10 = 0$, we are supposed to find all those values of X which satisfy this quadratic equation.
- ✓ By using quadratic formula or factoring we get x = 2 and x = 7 that satisfy the above equation.
- ✓ Let $a_n – 3\,a_{n-1} = 0$ for $n \geq 1$ be the recurrence relation
    - o We are aware that $A = \{a_n\}_{n=0}^{\infty}$is a function from the non-negative integers into the real numbers.
    - o Recurrence relation describes the relation between the values of the function at n and n – 1.
    - o Yes there is a function, defined with domain the set of non-negative integers, which makes the equation true for every value of n.
    - o It is shown as$A = \{a_n\}_{n=0}^{\infty}$ where $a_n = 3^n$ for $n \geq 0$.
    - o For this function we have $a_n – 3\,a_{n-1} = 3^n – 3(3^{n-1}) = 0$ for $n \geq 1$, so that this function satisfies the recurrence relation.
    - o There are possibilities of more solutions like if ***c*** is any constant the function $\{a_n\}_{n=0}^{\infty}$where $a_n = c3^n$ for $n \geq 0$ also satisfy the above recurrence relation as $a_n – 3\,a_{n-1} = c\,3^n – 3c\,(3^{n-1}) = 0$ for $n \geq 1$

CONCEPT: Suppose that S is a subset of the non-negative integers. Then a sequence A = $\{a_n\}_{n=0}^{\infty}$is a ***solution*** to a recurrence relation over S if the values a_n of A make the recurrence relation a true statement for every value of n in S. If the sequence $A = \{a_n\}_{n=0}^{\infty}$is a solution of a recurrence relation, then it is said to satisfy the relation.

Example: If $\mathbf{c_1}$ and $\mathbf{c_2}$are arbitrary constants, then $a_n = c_1 2^n + c_2 5^n$ satisfies the recurrence relation:

$a_n – 7 a_{n-1} + 10 a_{n-2} = 0$ over the set S of integers $n \geq 2$. Substituting this expression for a_n into the recurrence relation, we have

$a_n – 7 a_{n-1} + 10 a_{n-2} = (c_1 2^n + c_2 5^n) – 7 (c_1 2^{n-1} + c_2 5^{n-1}) + 10 (c_1 2^{n-2} + c_2 5^{n-2})$

$= c_1 2^n – 7c_1 2^{n-1} + 10 c_1 2^{n-2} + c_2 5^n – 7 c_2 5^{n-1} + 10 c_2 5^{n-2}$

$= c_1 2^{n-2} [2^2 – 7(2) + 10] + c_2 5^{n-2} [5^2 – 7(5) + 10]$

$= c_1 2^{n-2} (0) + c_2 5^{n-2} (0)$

$= 0$

- Boundary conditions are requirements that must be satisfied in addition to that of satisfying the recurrence relation.

Consider $a_n = c_1 2^n + c_2 5^n$, if we set n = 0 and n = 1, then $10 = a_0 = c_1 2^0 + c_2 5^0 = c_1 + c_2$ and $41 = a_1 = c_1 2^1 + c_2 5^1 = 2c_1 + 5c_2$.

Thus the constant c_1 and c_2 satisfies the equations

$$10 = c_1 + c_2 \qquad \text{and} \qquad 41 = 2c_1 + 5c_2$$

When we solve these two above equation we get $c_1 = 3$ and $c_2 = 7$.

Thus, $a_n = (3)2^n +(7) 5^n$is a solution of recurrence relation that satisfy the boundary conditions.

- If a linear recurrence relation of degree m has a constant coefficient then there will not be a unique solution.
- If there is some integer n_0 such that the values for $a_{n0}, a_{n1}, \ldots, a_{n0+m-1}$ are given then there will be a unique solution of the linear recurrence relation of degree m satisfying these boundary conditions.

Usually the values for $a_0, a_1, \ldots, a_{m-1}$ are given which is known as ***initial condition.***

7.5 THE FIBONACCI RELATION:

- Let us consider one problem to understand the Fibonacci relation. Suppose that there is one pair of squirrels, one male and one female, just born, and suppose, further, that every month each pair of squirrels (whose age is more than one month old) produces a new pair of offspring of opposite sexes. Find the number of squirrels after 12 months and after n months?
  - Firstly, we will consider one new born pair of squirrels.

Duration	Number of Pairs of Squirrel	Reason
After 1 month	ONE	As they are not mature to reproduce
After 2 month	TWO	Now the first pair has reproduced
After 3 months	THREE	Again the first pair has reproduced; second pair is yet to mature
After 4 months	FIVE	Again the First has reproduced; second pair has reproduced; third is yet to mature

- Thus, for each integer $n \geq 0$, let F_n denote the number of pairs of squirrels alive at the end of the nth month. Here $F_0 = 1$ as the original number of pairs of squirrels is one.

- We have $F_0 = 1 = F_1$, $F_2 = 2$, $F_3 = 3$ and $F_4 = 5$ from the above table.

- So F_n is formed by starting with F_{n-1} pairs of squirrels alive last month and adding the offspring's that can only come from the F_{n-2} pairs alive 2 months ago. So we have
  $F_n = F_{n-1} + F_{n-2}$ is the recurrence relation and $F_0 = F_1 = 1$ are the initial condition.

- Using the above relation and the values of $F_2 = 2$, $F_3 = 3$ and $F_4 = 5$, we get

$$F_5 = F_4 + F_3 = 5 + 3 = 8,$$
$$F_6 = F_5 + F_4 = 8 + 5 = 13$$
$$F_7 = F_6 + F_5 = 13 + 8 = 21$$
$$F_8 = F_7 + F_6 = 21 + 13 = 34$$
$$F_9 = F_8 + F_7 = 34 + 21 = 55$$
$$F_{10} = F_9 + F_8 = 55 + 34 = 89$$
$$F_{11} = F_{10} + F_9 = 89 + 55 = 144$$
$$F_{12} = F_{11} + F_{10} = 144 + 89 = 233.$$

So we have 233 pairs of squirrels alive after 12 months.

- The relation $F_n = F_{n-1} + F_{n-2}$ is called the ***Fibonacci relation.***

- The numbers F_n generated by the Fibonacci relation with the initial conditions $F_0 = F_1 = 1$ is called as ***Fibonacci numbers.***

- The sequence of Fibonacci numbers $\{F_n\}_{n=0}^{\infty}$ is the ***Fibonacci Sequence.***

7.5.1 Properties Of Fibonacci Numbers:

We will try to figure out the compact formula for the sum $S_n = F_0 + F_1 + \dots + F_n$. The following table below shows that $S_0 = 1 = F_2 - 2$; $S_1 = 2 = F_3 - 1$ and $S_2 = 4 = F_4 - 1$;

N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**
$\mathbf{F_n}$	1	1	2	3	5	8	13	21
$\mathbf{S_n}$	1	2	4	7	12	20	33	54

- This leads us to conjecture that:

1. The sum of the first n + 1 Fibonacci numbers is one less than F_{n+2}, that is $F_0 + F_1 + \dots + F_n = F_{n+2} - 1$.

   PROOF: Write the numbers in an array as follows:

$$F_0 = F_2 - F_1$$
$$F_1 = F_3 - F_2$$
$$F_2 = F_4 - F_3$$
$$F_3 = F_5 - F_4$$
$$\cdot \qquad \cdot$$
$$\cdot$$
$$F_n = F_{n+2} - F_{n+1}$$

   If we add all these equations, we will get $F_0 + F_1 + F_2 + \dots + F_n = F_{n+2} - F_1$. But $F_1 = 1$

2. $F_0 + F_2 + F_4 + \dots + F_{2n} = F_{2n+1}$

3. $F_0^2 + F_1^2 + \dots + F_n^2 = F_n F_{n+1}$

Notes

THEOREM: General Solution to the Fibonacci Relation.

If $\mathbf{F_n}$ satisfies the Fibonacci relation $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, then there are constants $\mathbf{C_1}$ and $\mathbf{C_2}$ such that

$$F_n = C_1\left(\frac{1+\sqrt{5}}{2}\right)^n + C_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

where the constants are completely determined by the initial conditions.

PROOF: Let $F(X) = \sum_{n=0}^{\infty} F_n X^n$ be the generating function for the sequence $\{F_n\}_{n=0}^{\infty}$. Then,

$$F(X) = F_0 + F_1X + F_2X^2 + F_3X^3 + \cdots + F_nX^n + \cdots$$
$$XF(X) = F_0X + F_1X^2 + F_2X^3 + \cdots + F_{n-1}X^n + \cdots$$
$$X^2F(X) = F_0X^2 + F_1X^3 + F_2X^4 + \cdots + F_{n-2}X^n + \cdots$$

Subtracting the last two equations from the first we will get the following results:

$$\begin{aligned} F(X) - XF(X) - X^2F(X) &= F_0 + (F_1 - F_0)X + (F_2 - F_1 - F_0)X^2 + (F_3 - F_2 - F_1)X^3 \\ &\quad + \cdots + (F_n - F_{n-1} - F_{n-2})X^n + \cdots \\ &= F_0 + (F_1 - F_0)X + 0X^2 + 0X^3 + \cdots \\ &= F_0 + (F_1 - F_0)X \end{aligned}$$

Thus,

$$F(X) = \frac{F_0 + (F_1 - F_0)X}{1 - X - X^2}$$

$$= \frac{F_0 + (F_1 - F_0)X}{[1 - \frac{(1+\sqrt{5})}{2}X][1 - \frac{(1-\sqrt{5})}{2}X]}$$

Thus, for whatever initial conditions on F_0 and F_1, the method of fraction applies to give

$$F(X)=\frac{C_1}{1-(1+\sqrt{5})X/2}+\frac{C_2}{1-(1-\sqrt{5})X/2}$$

Using the identities for geometric series we see that if $a=(1+\sqrt{5})/2$ and$b=(1-\sqrt{5})/2$ then,

$$F(X)=\frac{C_1}{1-aX}+\frac{C_2}{1-bX}=C_1\sum_{n=0}^{\infty}a^nX^n+C_1\sum_{n=0}^{\infty}b^nX^n$$

$$=\sum_{n=0}^{\infty}(C_1a^n+C_2b^n)X^n=\sum_{n=0}^{\infty}F_nX^n$$

In other words,

$F_n=C_1a^n+C_2b^n=C_1[(1+\sqrt{5})/2]^n+C_2[(1-\sqrt{5})/2]^n$for each n ≥ 0.

If we have given the initial condition we can find

$$C_1=\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)\ and\ C_2=\frac{-1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)$$

So in this case, the nth Fibonacci number is

$$F_n=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

Pascal's triangle states that states the sum of the elements which lie on the diagonal running upward from the left are Fibonacci numbers. We can illustrate this as:

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F_0	1			
F_1	1	1		
F_2	1	2	1	
F_3	1	3	3	1

F_4	1	4	6	4	1		
F_5	1	5	10	10	5	1	
F_6	1	6	15	20	15	6	1

We have the identity $F_n = C(n, 0) + C(n – 1, 1) + C (n – 2, 2) + \dots + C (n – k, k)$ where for $k = n / 2$, we find greatest integer in the selection.

To prove the above relation we define $q_n = C(n, 0) + C(n – 1, 1) + C (n – 2, 2) + \dots + C (n – k, k)$ for $n \geq 0$ and $k = \lfloor n / 2$

If we consider that $C(m , r) = 0$ for $r > m$, then we can write
$q_n = C(n, 0) + C(n – 1, 1) + C (n – 2, 2) + \dots + C (n – k, k) + C (n – k – 1, k + 1) + \dots + C (0, n)$.
We are supposed to prove that q_n satisfies the Fibonacci relation and that $q_0 = 1 = q_1$ but $q_0 = C(0, 0) = 1$ and $q_1 = C (1, 0) + C(0, 1) = 1$.
Using Pascal identity, we get following equation for $n \geq 2$.
$q_{n-1} + q_{n-2} = C (n – 1, 0) + C (n – 2, 1) + \dots + C (0, n – 1) + C (n – 2, 0) + C (n – 3, 1) + \dots + C (0, n – 2)$

$= C (n – 1, 0) + [C (n – 2, 1) + C (n – 2,0)] + [C (n – 3, 1) + C (n – 3,2)] + \dots + [C (0, n – 1) + C (0, n – 2)]$

$= C (n – 1, 0) + C (n – 1, 1) + C (n – 2, 2) + \dots + C (1, n – 1)$

$= C (n, 0) + C (n – 1, 1) + C (n – 2, 2) + \dots + C(1, n – 1) + C(0, n) = q_n$.

7.6 OTHER RECURRENCE RELATION MODEL:

Example 1: Let P represent the principal borrowed from a bank, let r equal the interest rate per period, and let a_n represent the amount due after n periods.
Then $a_n = a_{n-1} + r a_{n-1} = (1 + r) a_{n-1}$.

In particular $a_0 = P$, $a_1 = (1 + r) P$, $a_2 = (1 + r) a_1 = (1 + r)^2 P$ and so on, so that
$a_n = (1 + r)^n P$.

Example 2: Two armies engage in combat. Each army counts the number of men still in combat at the end of each day.

- Let a_0 and b_0 denote the number of men in the first and the second army, respectively, before the combat begins, and
- Let a_n and b_n denote the number of men in the two armies at the end of the nth day.
- Thus, $a_{n-1} - a_n$ represents the number of soldiers lost by the first army during the battle on the nth day.
- Similarly, $b_{n-1} - b_n$ represents the number of soldiers lost by the first army during the battle on the nth day.
- From above condition, we can conclude that
- Decrease in the number of soldiers in each army is proportional to the number of soldiers in the other army at the beginning of each day.
- So we have constants A and B such that $a_{n-1} - a_n = Ab_{n-1}$ and $b_{n-1} - b_n = B b_{n-1}$
- These constants measure the effectiveness of the weapons of the different armies.
- We can also rewrite this as below:
  $a = a_{n-1} - Ab_{n-1}$ and $b = - Ba_{n-1} + b_{n-1}$
  which represents very much reminiscent of two –linear equation in two unknowns.

Example 3: Find a recurrence relation for the number of n-digit ternary sequences that have an even number of 0's.

- If the first digit is not 0, then there are $2 a_{n-1}$ $(n - 1)$ digit such ternary sequences. If the first digit is 0, then we must count the number of $(n - 1)$- digit ternary sequences that have an odd number of 0's.

Notes

- Since there are 3^{n-1} total (n – 1)- digit ternary sequences and a_{n-1} with an even number of 0's and that start with 0.
- Thus, $a_n = 2\, a_{n-1} + 3^{n-1} - a_{n-1} = a_{n-1} + 3^{n-1}$

Example 4: Suppose a coin is flipped until 2 heads appear and then the experiment stops. Find a recurrence relation for the number of experiments that end on the nth flip or sooner.

- $a_n = a_{n-1} + (n-1)$

7.7 SOLVING RECURRENCE RELATION:

We will consider the following example to clarify the different methods.

1. Solve the recurrence relation $a_n = a_{n-1} + f(n)$ for $n \geq 1$ by substitution.

$$a_1 = a_0 + f(1)$$
$$a_2 = a_1 + f(2) = [a_0 + f(1)] + f(2)$$
$$a_3 = a_2 + f(3) = [a_0 + f(1) + f(2)] + f(3)$$
$$\vdots$$
$$a_n = a_0 + f(1) + f(2) + \ldots + f(n)$$

$$= a_0 + \sum_k^{\infty} f(k)$$

Moreover generally, if c is a constant then we can solve $a_0 = ca_{n-1} + f(n)$ for $n \geq 1$ in the same way:

$$a_1 = c\, a_0 + f(1)$$
$$a_2 = c\, a_1 + f(2) = c\,[c\, a_0 + f(1)] + f(2)$$
$$= c^2 a_0 + c\, f(1) + f(2)$$
$$a_3 = c\, a_2 + f(3) = c\,[c^2 a_0 + c\, f(1) + f(2)] + f(3)$$
$$= c^3 a_0 + c^2 f(1) + c\, f(2) + f(3)$$
$$\vdots$$

.

$$\mathbf{a_n = c\,a_{n-1} + f\,(n) = c[\,c^{n-1}\,a_0 + c^{n-2}\,f\,(1) + \ldots + c\; f\,(n-2) + f\,(n-1) + f\,(n\,)}$$

$= c^{n-1}a_0 + c^{n-1} f\,(1) + c^{n-2} f\,(2) + \ldots + c\, f\,(n - 1) + f\,(n\,)$

Or

$$a_n = c^n a_0 + \sum_{k=1}^{n} c^n f(k)$$

7.7.1 Solution By Generating Functions:

First we will try to understand the shifting properties of generating functions:

- If $A(X) = \sum_{n=0}^{\infty} a_n X^n$ generates the sequence (a_0, a_1, a_2, …),
- Then X A(X) generates the sequence (0, a_0, a_1, a_2, …);
- X^2 A(X) generates the sequence (0,0, a_0, a_1, a_2, …);
- In general, X^k A(X) generates the sequence (0, 0…0,a_0, a_1, a_2…) where there are k zeros before a_0.
- If A(X) is the generating function for the sequence (a_0, a_1, a_2…), then multiply A(X) by X amounts to shifting the sequence one place to the right and inserting a zero in front.
- Similarly if we multiply by X^k amounts to shifting the sequence k positions to the right and inserting k zeros in front.
- The above process can be demonstrated in the formal power series expression as follows:

$$X^k A(X) = X^k \sum_{n=0}^{\infty} a_n X^n = X^k \sum_{n=0}^{\infty} X^{n+k}$$

- If we replace **n + k** by **r** and then we will have **n = r – k** so we have new equation after substitution as $\sum_{r=k}^{\infty} a_{r-k}X^r$ which signifies that it generates the sequence
  $\{b_r\}_{r=0}^{\infty}$where
  $0 = b_0 = b_1 = \ldots = b_{k-1}$, $b_k = a_0$, $b_{k+1} = a_1$ and in general $b_r = a_{r-k}$ if $r \geq k$.
  Thus, the nth term in the new sequence is obtained from the old sequence by replacing
  $a_n = a_{n-k}$ if $n \geq k$ and by 0 if $n < k$.
- For instance, we know that $1 / (1 – X) = \sum_{n=0}^{\infty} X^n$generates the sequence (1, 1, 1, **…**) , that is, the sequence $\{a_n\}_{n=0}^{\infty}$where $a_n = 1$ for each $n \geq 0$.
- Thus,

$$\frac{X}{1-X} = \sum_{n=0}^{\infty} X^{n+1} = \sum_{r=1}^{\infty} X^r$$

Generates (0, 1, 1, 1, …) and

$$\frac{X^2}{1-X} = \sum_{n=0}^{\infty} X^{n+2} = \sum_{r=2}^{\infty} X^r$$

Generates (0,0, 1, 1, 1, …). Similarly,

$$\frac{1}{(1-X)^2} = \sum_{n=0}^{\infty} C(n+1,n)X^n = \sum_{r=2}^{\infty}(n+1)X^n$$

Generates the sequence (1, 2, 3, 4 …) so that

$$\frac{X}{(1-X)^2} = \sum_{n=0}^{\infty}(n+1)X^{n+1} = \sum_{r=1}^{\infty} rX^r$$

Generates the sequence $\{r\}_{r=0}^{\infty}$= (0, 1, 2, 3, 4…).

[$\sum_{r=1}^{\infty} rX^r$ describes the coefficient of X^0 is 0 because the sum is taken from r = 1 to ∞, but still we get same conclusion]

So we can write

$$\frac{X}{(1-X)^2} = \sum_{r=1}^{\infty} rX^r \; and \; as = \sum_{r=0}^{\infty} rX^r$$

Both the above expression indicates that the coefficient of X^0 is zero.

Similarly,

$$\frac{X^2}{(1-X)^2} = \sum_{n=0}^{\infty} (n+1)X^{n+2} = \sum_{r=2}^{\infty} (r-1)X^r$$

Generates the sequence (0, 0, 1, 2, 3, 4…) that is the sequence $\{b_r\}_{r=0}^{\infty}$where $b_r = r - 1$ if $r \geq 2$ but $0 = b_0 = b_1$. Since, the expression $b_r = r - 1$ equal to zero when r = 1, we can write the expression $\frac{X^2}{(1-X)^2} = \sum_{r=2}^{\infty}(r-1)X^r$ as $\sum_{r=1}^{\infty}(r-1)X^r$.

Following the above procedure,

$$\frac{1}{(1-X)^3} = \sum_{n=0}^{\infty} C(n+2,n)X^n = \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2} X^n$$

generates the sequence

$$\left\{\frac{(n+2)(n+1)}{2}\right\}_{n=0}^{\infty} = \left(\frac{1.2}{2}, \frac{2.3}{2}, \frac{3.4}{2}, \dots\right),$$

and therefore,

$$\frac{2}{(1-X)^3} = \sum_{n=0}^{\infty} (n+2)(n+1)X^n$$

generates

$$\{(n+2)(n+1)\}_{n=0}^{\infty} = (1.2, 2.3, 3.4, \dots).$$

But then,

$$\frac{2X}{(1-X)^3} = \sum_{n=0}^{\infty} (n+2)(n+1)X^{n+1} = \sum_{r=1}^{\infty} (r+1)(r)X^r$$

generates the sequence (0, 1.2,2.3,3.4,…).

Now since $b_r = (r+1)(r)$ equals 0 where r = 0, we can write

$$\frac{2X}{(1-X)^3} = \sum_{r=1}^{\infty}(r+1)(r)X^r = \sum_{r=0}^{\infty}(r+1)(r)X^r$$

so that $\frac{2X}{(1-X)^3}$ generates $\{(r+1)(r)\}_{r=0.}^{\infty}$

Similarly,

$$\frac{2X^2}{(1-X)^3} = \sum_{n=0}^{\infty}(n+2)(n+1)X^{n+2} = \sum_{r=2}^{\infty}(r)(r-1)X^r$$

$$= \sum_{r=0}^{\infty}(r)(r-1)X^r$$

generates the sequence (0,0,1.2,2.3,3.4,…) and the last sum can be taken from 0 to ∞ because the coefficient r (r – 1) is 0 when r = 0, 1. In this way, we can combine these results to obtain generating functions for other sequences.

For example,

$$\frac{2X}{(1-X)^3} - \frac{X}{(1-X)^2} = \frac{X(1+X)}{(1-X)^3}$$

generates the sequence $\{(r+1)(r) - r\}_{r=0.}^{\infty} = \{r^2\}_{r=0.}^{\infty} =$ $(0,1,4,9,\ldots)$

In the similar way,

$$\frac{1}{(1-X)^4} = \sum_{n=0}^{\infty} C(n+3,n)X^n = \sum_{n=0}^{\infty}\frac{(n+3)(n+2)(n+1)}{6}X^n$$

generates $\left\{\frac{(n+3)(n+2)(n+1)}{6}\right\}_{n=0}^{\infty}$; $\frac{6}{(1-X)^4}$ generates $\{(n+3)(n+2)(n+1)\}_{n=0}^{\infty}$;

$$\frac{6X}{(1-X)^4} = \sum_{n=0}^{\infty}(n+3)(n+2)(n+1)X^{n+1}$$

$$= \sum_{r=1}^{\infty}(r+2)(r+1)(r)X^r$$

$$= \sum_{r=0}^{\infty}(r+2)(r+1)(r)X^r$$

generates $\{(r+2)(r+1)(r)\}_{r=0.}^{\infty}$; *and*

$$\frac{6X^2}{(1-X)^4} = \sum_{n=0}^{\infty}(n+3)(n+2)(n+1)X^{n+2}$$
$$= \sum_{r=1}^{\infty}(r+1)(r)(r-1)X^r$$
$$= \sum_{r=0}^{\infty}(r+1)(r)(r-1)X^r$$

generates $\{(r+1)(r)(r-1)\}_{r=0.}^{\infty}$

Since $(r+3)(r+2)(r+1) = r^3 + 6r^2 + 11r + 6$ then $r^3 = (r+3)(r+2)(r+1) - 6r^2 - 11r - 6$ so that $\{r^3\}_{r=0}^{\infty}$is generated by

$$\frac{6}{(1-X)^4} - \frac{6(X)(1+X)}{(1-X)^3} - 11\frac{X}{(1-X)^2} - \frac{6}{(1-X)} = \frac{X(1+4X+X^2)}{(1-X)^4}$$

In similar way we can find the generating functions for the sequences$\{r^4\}_{r=0}^{\infty}, \{r^5\}_{r=0}^{\infty}$and so on.

2. If $A(X) = \sum_{n=0}^{\infty} a_n X^n$ generates the sequence (a_0, a_1, a_2, **…**), then

   $A(X) - a_0 = \sum_{n=1}^{\infty} a_n X^n$ generates the sequence (0, a_1, a_2, **…**) and in general

   $A(X) - a_0 - a_1X - \cdots - a_{k-1}X^{k-1} = \sum_{n=k}^{\infty} a_n X^n$generates(0, 0, **…,**0, a_k**,** a_{k+1}**…),** where there are **k** zeros before a_k**.**But when we divide it by powers of X shifts the sequence to the left like for example, $\frac{A(X)-a_0}{X} = \sum_{n=1}^{\infty} a_n X^{n-1}$ generates the sequence $(a_1, a_2, a_3, \dots)$; $\frac{A(X)-a_0-a_1X}{X^2}$ generates $(a_2, a_3, a_4, \dots)$; and inn general for $k \geq 1$,

   $\frac{A(X)-a_0-a_1X-\cdots-a_{k-1}X^{k-1}}{X^k}$ generates$(a_k, a_{k+1}, a_{k+2}, \dots)$.

Notes

Similarly, if we substitute n – k = r and the expression

$\frac{A(X)-a_0-a_1X-\cdots-a_{k-1}X^{k-1}}{X^k} = \sum_{n=k}^{\infty} a_n X^{n-k}$ becomes

$\sum_{r=0}^{\infty} a_{r+k} X^r$ which in the original sequence is replaced by a_{n+k} for each n, indicating that the sequence has been shifted **k** places to the left.

Example 1: Solve the recurrence relation

$a_n - 7a_{n-1} + 10a_{n-2} = 0$ for $n \geq 0$

We will number the steps of the procedure.

1. Let $A(X) = \sum_{n=0}^{\infty} a_n X^n$.

2. Next multiply each term in the recurrence relation by X^n and sum from 2 to ∞:

$$\sum_{n=2}^{\infty} a_n X^n - 7\sum_{n=2}^{\infty} a_{n-1} X^n + 10\sum_{n=2}^{\infty} a_{n-2} X^n = 0$$

3. Replace each infinite sum by an expression from the table of equivalent expressions:

$[A(X) - a_0 - a_1X] - 7X[A(X) - a_0] + 10X^2[A(X)]$

4. Then simplify:

$A(X)(1 - 7X + 10X^2) = a_0 + a_1X - 7a_0X$ or

$$A(X) = \frac{a_0 + (a_1 - 7a_0)X}{1 - 7X + 10X^2} = \frac{a_0 + (a_1 - 7a_0)X}{(1 - 2X)(1 - 5X)}$$

5. Decompose A(X) as a sum of partial fractions:

$$A(X) = \frac{C_1}{1 - 2X} + \frac{C_2}{1 - 5X}$$

where C_1 and C_2 are constants, as yet undetermined.

6. Express A(X) as a sum of familiar series:

$$A(X) = \frac{C_1}{1-2X} + \frac{C_2}{1-5X} = C_1 \sum_{n=0}^{\infty} 2^n X^n + C_2 \sum_{n=0}^{\infty} 5^n X^n$$

7. Express a_n as the coefficient of X^n in A(X) and in the sum of the other series:

$$a_n = C_1 2^n + C_2 5^n$$

8. Now the constants C_1 and C_2 are uniquely determined once values for a_0 and a_1 are given. For example, if $a_0 = 10$ and $a_1 = 41$, we may use form of the general solution $a_n = C_1 2^n + C_2 5^n$, and let n = 0 and n = 1 to obtain the equations

$C_1 + C_2 = 10$ and $2C_1 + 5C_2 = 41$,

which determine the values $C_1 = 3$ and $C_2 = 7$.

The unique solution of the recurrence relation is $a_n = (3)2^n + (7)5^n$

THEOREM: If $\{a_n\}_{n=0}^{\infty}$ is a sequence of numbers which satisfy the linear recurrence relation with constant coefficients $a_n + c_1 a_{n-1} + \ldots + c_k a_{n-k} = 0$, where $c_k \neq 0$, and $n \geq k$, then the generating function $A(X) = \sum_{n=0}^{\infty} a_n X^n$ equals P(X) / Q(X), where $P(X) = a_0 + (a_1 + c_1 a_0) X + \ldots + (a_{k-1} + c_1 a_{k-2} + \ldots + c_{k-1} a_0) X^{k-1}$ and
$Q(X) = 1 + c_1 X + \ldots + c_k X^k$.

Conversely, If P(X) and Q(X) are polynomials given, where P(X) has degree less than k, there is a sequence $\{a_n\}_{n=0}^{\infty}$ whose generating function is A(X) = P(X) / Q(X).

Notes

The sequence $\{a_n\}_{n=0}^{\infty}$ satisfies a linear homogenous recurrence relation with constant coefficients of degree k, where the coefficients of the recurrence relation are the coefficients of Q(X).

In fact, if $Q(X) = b_0 + b_1 X + \ldots + b_k X^k$ where $b_0 \neq 0$ and $b_k \neq 0$ then

$$Q(X) = b_0(1 + \frac{b_1}{b_0}X + \cdots + \frac{b_k}{b_0}X^k)$$
$$= b_0(1 + c_1X + \cdots + c_kX^k)$$

where $c_i = b_i / b_0$ for $i \geq 1$. Then

$$A(X) = \frac{P(X)}{Q(X)} = \frac{\frac{1}{b_0}P(X)}{1 + c_1X + \cdots c_kX^k}$$

and the coefficient of A(X) are discovered by using partial fractions and the factors of

$1 + c_1 X + \ldots + c_k X^k$. then the recurrence relation satisfied by the coefficients of A(X) is

$a_n + c_1 a_{n-1} + \ldots + c_k a_{n-k} = 0$.

Example: Solve the following recurrence relations using generating functions.

a. $a_n - 9a_{n-1} + 20a_{n-2} = 0$ for $n \geq 2$ and $a_0 = -3$, $a_1 = -10$
b. $a_n - 5a_{n-1} + 6a_{n-2} = 0$ for $n \geq 2$ and $a_0 = 1$, $a_1 = -2$
c. $a_n - 3a_{n-2} + 2a_{n-3} = 0$ for $n \geq 3$ and $a_0 = 1$, $a_1 = 1$, $a_2 = 2$.
d. $a_n + a_{n-1} - 16 a_{n-2} + 20a_{n-3} = 0$ for $n \geq 3$ and $a_0 = 1$, $a_1 = 1$, $a_2 = -1$

Solution:

a. $a_n = 2.5^n - 5.4^n$
b. $A(X) = [1 - 7X] / [1 - 5X + 6X^2]$
   $= [5 / 1 - 2X] - [4 / 1 - 3X]$;
   $a_n = 5(2^n) - 4(3^n)$
c. $a_n = 8/9 - 6/9\, n + 1/9(-2)^n$
d. $A(X) = [X / 1 + X - 16X^2 + 20X^3]$

$$= \frac{-2/49}{1-2X} + \frac{7/49}{(1-2X)^2} - \frac{5/49}{1+5X}$$

$$a_n = -2/49(2^n) + 7/49(n+1)(2^n) - 5/49(-5)^n$$

7.7.3 The Method Of Characteristic Roots:

- Consider the example 1 of previous section where the denominator $Q(X) = 1 - 7X + 10X^2$ and the general solution for a_n was $a_n = C_1 2^n + C_2 5^n$ because Q(X) factors as $(1 - 2X)(1 - 5X)$. Note that the roots of Q(X) were 1 / 2 and 1 / 5 while the solutions involve powers of their reciprocals.
- To avoid this reciprocal relationship, let us consider another polynomial where we will replace X in Q(X) by t^2 to obtain the polynomial

$$C(t) = t^2 Q(1/t) = t^2 [1 - 7(1/t) + 10(1/t)^2]$$
$$= t^2 - 7t + 10$$
$$= (t-2)(t-5)$$

  Thus, the roots of the above polynomial , 2 and 5 are in direct relationship with the form of the solution for $a_n = C_1 2^n + C_2 5^n$.
- The polynomial C(t) is known as the characteristics polynomial of the recurrence relation.
- If the recurrence relation is $a_n + c_1 a_{n-1} + \ldots + c_k a_{n-k} = 0$ for $n \geq k$, where $c_k \neq 0$, then the characteristics polynomial for this recurrence relation is

  $C(t) = t^k + c_1 t^{k-1} + \ldots + c_k$ and this in turn equals to $t^k Q(1/t)$, where $Q(X) = 1 + c_1 X + \ldots + c_k X^k$. then if C(t) factors as $(t - \alpha_1)^{r1} \ldots (t - \alpha_s)^{rs}$, then in the expression A(X) = P(X) / Q(X), the denominator Q(X) factors as $(1 - \alpha_1 X)^{r1} \ldots (1 - \alpha_s X)^{rs}$

DISTINCT ROOTS:

If the characteristics polynomial has distinct roots $\alpha_1 \ldots \alpha_k$ **,** then general form of the solutions for the homogenous equation is $a_n = C_1$ n n

α_1 + …+ $C_k\ \alpha_k$ where C_1, C_2 , … , C_k are constants which may be chosen to satisfy any initial conditions.

Example: To solve $a_n - 7\,a_{n-1} + 12\,a_{n-2} = 0$ for $n \geq 2$, the characteristics equation is

$C(t) = t^2 - 7t + 12 = (t - 3)(t - 4)$.

Thus, the general solution is $a_n = C_1\, 3^n + C_2\, 4^n$.

If the initial condition are $a_0 = 2$, $a_1 = 5$, then we should follow to the steps below:

$$C_1 + C_2 = 2 \text{ and } 3C_1 + 4C_2 = 5$$

So we get $C_1 = 3$ and $C_1 = -1$, and the required solution is

$$a_n = (3)\, 3^n - 4^n.$$

7.7.4 Distinct Roots &Multiple Roots:

Example: Write the general form of the solutions to

a. $a_n - 6\,a_{n-1} + 9\,a_{n-2} = 0$

b. $a_n - 3\,a_{n-1} + 3\,a_{n-2} - a_{n-3} = 0$

c. $a_n - 9\,a_{n-1} + 27\,a_{n-2} - 27\,a_{n-3} = 0$

- ✓ Since the characteristics polynomial in (a) is $t^2 - 6t + 9 = (t - 3)^2$ the general solution in the form $a_n = D_1\, 3^n + D_2\, n\, 3^n$.
- ✓ Likewise the characteristic polynomial for (b) is $t^3 - 3t^2 + 3t - 1 = (t - 1)^3$ so the general solution is $a_n = D_1 + D_2\, n + D_3\, n^2$
- ✓ In (c) the characteristic polynomial is $t^3 - 9t^2 + 27t - 27 = (t - 3)^3$ then the general solution is $a_n = D_1\, 3^n + D_2\, n 3^n + D_3\, n^2\, 3^n$.

THEOREM:

Let the distinct roots of the characteristic polynomial , $C(t) = t^k + c_1 t^{k-1} + \ldots + c_k$ of the linear homogenous recurrence relation, $a_n + c_1\, a_{n-1} + \ldots + c_k\, a_{n-k} = 0$,where $n \geq k$ and $c_k = 0$ be

$\alpha_1, \alpha_2, \ldots, \alpha_s$ where $s \le k$. Then there is a general solution for a_n which is in the form, $U_1(n) + U_2(n) + \ldots + U_s(n)$ where $U_i(n) = (D_{i0} + D_{i0}n + D_{i0}n^2 + \ldots + D_{im-1}\ n^{mi-1})\alpha_i^n$

and where m_i is the multiplicity of the root α_i.

EXAMPLE:

1. Suppose that the characteristic polynomial for a linear homogenous recurrence relation is $(t-2)^3(t-3)^2(t-4)^3$. Then the general solution is $a_n = (D_1 + D_2 n + D_3 n^2)\ 2^n + (D_4 + D_5 n)\ 3^n + (D_1 + D_2 n + D_3 n^2)\ 4^n$.
2. Do the same for the recurrence relation $a_n - 5a_{n-1} + 8a_{n-2} - 4a_{n-3} = 0$ for
   $n \ge 3$.
   $C(t) = t^3 - 5t^2 + 8t - 4$
   $= (t-2)^2(t-1)$
3. What is the solution to the recurrence relation $an = 2a^{n-1} + 3a^{n-2}$, with $a_0 = 3$ and $a_1 = 5$?

The characteristic equation for this recurrence is $0 = r^2 - 2r - 3 = (r-3)(r+1)$, which has roots $r_1 = 3$ and $r_2 = -1$.

Now we have a solution in the form $an = d_1 3n + d_2(-1)n$, for

Some d_1 and d_2.

We can find the constants from the initial values we know:

$$a0 = d_1 3^0 + d_2(-1)^0 = d_1 + d_2 = 3,$$
$$a1 = d_1 3^1 + d_2(-1)^1 = 3d_1 - d_2 = 5.$$

Adding these equations, we get $4d_1 = 8$, so $d_1 = 2$. And then from the first equation, we have $d_2 = 1$.

Finally, we have a solution: $a_n = 2\cdot 3^n + (-1)^n$.

- We can calculate the first few terms, either with the recurrence or the solution: 3, 5, 19, 53, 163, 485.

Check Your Progress 3

1. What is Fibonnaci Series and state its properties

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2. Explain the Generating function method

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7.8 LET'S SUM UP

Generating function is used mostly to describe the infinite sequence of numbers. They are widely used to find a closed formula for sequence in recurrence relation, find relationship between sequences, solve Combinatory problems, proving identities involving sequence.

7.9 KEYWORDS

1. Expression : is a finite combination of symbols that is well-formed according to rules that depend on the context.

2. Quadratic Equation: is a second-order polynomial equation in a single variable

3.Geometric series- is a series with a constant ratio between successive terms.

4. Initial condition - in some contexts called a seed value, is a value of an evolving variable at some point in time designated as the initial time (typically denoted t = 0).

7.10 QUESTION FOR REVIEW

1. Find the generating function for a_r =the number of ways of distributing r similar balls into 7 numbered boxes where the second, third, fourth and fifth boxes are non-empty.
2. Find the coefficient of X^{20} in $(X^3 + X^4 + X^5 \ldots)^5$
3. Find a recurrence relation for a_n the number of different ways to distribute either a Rs. 1 bill, Rs. 2 bill, a Rs. 5 bill or Rs. 10 bill on a successive days until a total of n rupees has been distributed.
4. Explain the analysis of the Mergesort Algorithm
5. Solve $a_n - 6a_{n-1} + 12a_{n-2} - 8a_{n-3} = 0$ by generating function.

7.11 SUGGESTED READINGS

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7.12 ANSWER TO CHECK YOUR PROGRESS

1. Explain the concept with example – 7.1
2. Provide definition –7.1.1
3. Explain the concept of Linear Recurrence Relation --- 7.3
4. Explain the concept--- 7.3.1
5. Explain the concept --- 7.5 & 7.5.1
6. Explain the steps--- 7.7.2